BOUNDING TAIL PROBABILITIES IN DYNAMIC ECONOMIC MODELS

JOHN STACHURSKI

ABSTRACT. This paper provides conditions for bounding tail probabilities in stochastic economic models in terms of their transition laws and shock distributions. Particular attention is given to conditions under which the tails of stationary equilibria have exponential decay. By way of illustration, the technique is applied to a threshold autoregression model of exchange rates.

1. INTRODUCTION

This paper provides bounds on probabilities of tail events in terms of model primitives. By definition, tail events occur only infrequently, but their impact can be large. A classic example is fluctuations in asset prices. For example, the stock market crash on 19th October 1987 saw the Dow Jones Industrial Average drop by 22% in one day, eliminating nearly US$1 trillion in market capitalization. The financial crisis that engulfed many Asian economies in the middle of 1997 likewise led to sharp devaluations in the exchange rates of several Asian currencies, with far-reaching economic consequences.

Thus, one application of our results is in the modeling of financial variables. As observed by Mandelbrot (1963), heavy tails are found in many kinds of market returns data.[1] The property of having heavy tails is often associated with "chaotic" or highly nonlinear behavior in the model which describes motion of the system (see, for example, Lux, 1998; or Pellicer-Lostao and López-Ruiza, 2010). One of

The author thanks without implication Vance Martin, Lawrence Uren and Makoto Yano for many helpful comments, as well as the Australian Research Council for its financial support.

[1] For a more recent overview of the literature see Rachev (2001).

the contributions of this paper is to show that a large class of highly nonlinear and discontinuous models in fact generate marginal and stationary distributions with exponentially decreasing tails. As a result, these models can *not* represent time series which empirically are observed to feature heavy tails.

Another potential application of this research is when the state variable is itself a distribution. For example, it often happens that in macroeconomic dynamics one wishes to study a situation where each entity in a given economic model has a vector of endogenously evolving attributes, such as income, wealth, asset holdings, human capital, wage rate, and so on. The state of the economy is given by the distribution of these attributes across the population. In this case, the size of the distribution tails provides a measure of dispersion.

Our focus is on the broad class of economic models that can be represented as time-homogeneous Markov chains, with discrete time parameter and continuous state spaces. The methodology developed here is based on a generalized "drift condition". Our condition complements the more standard affine drift conditions used extensively in the existing literature to establish stability, stationarity and ergodicity of stochastic processes.[2] It is the source of the tail bounds derived in the paper.

Previously, Borovkov (1998, Theorem 3.1) also studied bounds on the tails of the marginal distributions of Markov chains. His bounds are not directly comparable with those given here. The main difference is in the conditions on the primitives used to derive the bounds. Our technique is intended to fit the kind of equilibrium structure typically available in economic models. For example, in our exchange rate application, the drift is due to arbitrage, which pushes the rate towards its purchasing power parity equilibrium.

Section 2 formulates the problem. Section 3 sets out the drift condition and derives some of its immediate consequences. Section 4 gives a number of applications which illustrate the method.

[2]See, for example, Meyn and Tweedie (2009), or Borovkov (1998).

2. Formulation of the Problem

Consider an economic process, the state vector of which takes values in space S, a Borel subset of $\mathbb{R}^n$. The law of motion is given by

$$X_{t+1} = h(X_t, \xi_{t+1}), \quad X_0 = x_0 \in S, \quad \{\xi_t\}_{t=0}^{\infty} \overset{\text{IID}}{\sim} \varphi. \tag{1}$$

The vectors $\{X_t\}$ all take values in S, the shocks ξ_t take values in Z, a Borel subset of $\mathbb{R}^k$, and h is a measurable function mapping $S \times Z \to S$. The shocks are generated on probability space $(\Omega, \mathscr{F}, \mathbf{P})$, and $\mathbf{E}$ is the expectations operator corresponding to $\mathbf{P}$.[3]

For topological space T, we let $\mathscr{B}(T)$ denote the Borel sets, and $\mathscr{P}(T)$ denote the probability measures on $(T, \mathscr{B}(T))$. The common distribution of ξ_t is denoted by $\varphi \in \mathscr{P}(Z)$, while that of X_t is denoted by $\psi_t \in \mathscr{P}(S)$. Also, $\mathbb{1}_B$ is the indicator function of B. Thus, for example, $\mathbf{E}\, \mathbb{1}_B \circ X_t = \psi_t(B)$ holds for every $B \in \mathscr{B}(S)$.

Given elements μ and ν in $\mathscr{P}(T)$, their total variation distance is defined as

$$\|\mu - \nu\|_{TV} := \sup_{B \in \mathscr{B}(T)} |\mu(B) - \nu(B)|.$$

For $\{\mu_n\}_{n=0}^{\infty} \subset \mathscr{P}(T)$ and $\mu \in \mathscr{P}(T)$ we say that μ_n converges to μ if $\|\mu_n - \mu\|_{TV} \to 0$ as $n \to \infty$. If $\{X_n\}_{n=0}^{\infty}$ and X are T-valued random variables, we say that X_n converges to X if the distribution of X_n converges to that of X.[4]

---

[3] In time series modeling and macroeconomic dynamics it is common to deal with seemingly more complex models than (1). For example, X_{t+1} might depend on $X_t, \ldots, X_{t-j}$ for some j, and the shocks might themselves be correlated of some finite order. However, such models can always be rewritten in the form of (1) by suitably expanding the number of state variables. As a result, in all of what follows we concentrate only on models with this simple first order representation (1).

[4] Convergence in total variation is stronger than convergence in distribution in the usual sense. See, for example, Stokey, Lucas and Prescott (1989, Chapters 10–11).

We also define stationary distributions and ergodicity. A probability $\psi^* \in \mathscr{P}(S)$ is called *stationary* for (1) iff

$$\int \left[\int \mathbb{1}\{h(x,z) \in B\} \varphi(dz) \right] \psi^*(dx) = \psi^*(B), \quad \forall B \in \mathscr{B}(S).$$

If the current (i.e., time t) distribution is ψ^*, then the left hand side gives the probability that $X_{t+1} \in B$. Thus, if ψ^* satisfies this equation, then this probability is $\psi^*(B)$, which is the same as it is today. Since this holds for all B, we have $\psi_t = \psi_{t+1} = \psi^*$.

The process (1) is called *ergodic* if it has a unique stationary distribution $\psi^* \in \mathscr{P}(S)$, and, in addition, ψ_t converges to ψ^* for every $x_0 \in S$. It is *geometrically ergodic* if, moreover, $\|\psi_t - \psi^*\|_{TV} = O(\rho^t)$ for some $\rho < 1$.

3. A Drift Condition

We begin with a drift condition that can be used to bound the tails of the marginal distributions ψ_t, and of the stationary distribution ψ^* when it exists. To state the condition, let $w \colon S \to \mathbb{R}_+$ be a given measurable function. To the extent that $w(x)$ converges rapidly to infinity as $\|x\| \to \infty$, bounds on $\mathbf{E} w(X_t)$ restrict the tails of the distribution ψ_t. For example, if $w(x) = e^{\|x\|}$, then Chebychev's inequality yields

$$\mathbf{P}\{\|X_t\| > r\} = \mathbf{P}\{e^{\|X_t\|} > e^r\} \leq e^{-r}\, \mathbf{E} e^{\|X_t\|} = e^{-r}\, \mathbf{E} w(X_t). \tag{2}$$

One implication is that if $\mathbf{E} w(X_t) = \mathbf{E} e^{\|X_t\|}$ is finite, then we have $\mathbf{P}\{\|X_t\| > r\} = O(e^{-r})$, and ψ_t has exponentially decreasing tails.

Taking $w \colon S \to \mathbb{R}_+$ as given, we introduce the following condition on w and the process (1).

Condition 3.1. There exists an increasing concave function $\kappa \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int w[h(x,z)] \varphi(dz) \leq \kappa[w(x)] \quad \text{for all } x \in S.$$

This drift condition is a generalization of the standard drift condition used in the Markov process literature (see, e.g., Meyn and Tweedie, 2009), where κ is an affine function with slope less than one.

Example 3.1. Consider a one-sector optimal growth model with savings function σ. Suppose for simplicity that depreciation is total between periods, and capital stock evolves according to the rule

$$k_{t+1} = h(k_t, \xi_{t+1}) := \sigma[f(k_t, \xi_{t+1})],$$

where f is a production function and $\{\xi_t\}$ is an IID sequence of productivity shocks with distribution φ. As is conventional, we assume that f is concave and increasing in its first argument. Seeking a bound on the first moment, we take $w(x) = x$. If the agent cannot borrow then savings is limited by current income, and $\sigma(x) \leq x$. In this case, we have

$$\int w[h(k,z)]\varphi(dz) = \int \sigma[f(k,z)]\varphi(dz) \leq \int f(k,z)\varphi(dz).$$

Defining $\kappa(x) = \int f(x,z)\varphi(dz)$, this becomes

$$\int w[h(k,z)]\varphi(dz) \leq \kappa(k) = \kappa[w(k)].$$

Since k is arbitrary and κ is concave and increasing, we see that Condition 3.1 is satisfied.

Using Condition 3.1, we can state the following proposition.

Proposition 3.1. *Let $\{X_t\}_{t=0}^{\infty}$ be the S-valued stochastic process defined by* (1)*, and let $w \colon S \to \mathbb{R}_+$ be given. If Condition 3.1 holds, then*

$$\mathbf{E}\, w(X_t) = \int w d\psi_t \leq \kappa^t[w(x_0)] \qquad (t \in \mathbb{N}).$$

Here κ^t is the t-th composition of κ with itself.

Proof of Proposition 3.1. Let $\{\mathscr{F}_t\}_{t=0}^{\infty}$ be the natural filtration for $\{\xi_t\}_{t=0}^{\infty}$, and fix any $t \in \mathbb{N}$. By definition,

$$\mathbf{E}\,[w \circ X_{t+1} \mid \mathscr{F}_t] = \mathbf{E}\,[w \circ h(X_t, \xi_{t+1}) \mid \mathscr{F}_t].$$

Since X_t is $\mathscr{F}_t$-measurable and ξ_{t+1} is independent of $\mathscr{F}_t$, we obtain

$$\mathbf{E}\left[w \circ X_{t+1} \mid \mathscr{F}_t\right] = \int w \circ h(X_t, z)\, \varphi(dz).$$

Applying Condition 3.1, we then have the bound

$$\mathbf{E}\left[w \circ X_{t+1} \mid \mathscr{F}_t\right] \leq \kappa \circ w \circ X_t \quad \mathbf{P}\text{-almost surely.}$$

$$\therefore \quad \mathbf{E}\left[w \circ X_{t+1}\right] \leq \mathbf{E}\left[\kappa \circ w \circ X_t\right].$$

Using concavity of κ and Jensen's inequality yields $\mathbf{E}\left[w \circ X_{t+1}\right] \leq \kappa\{\mathbf{E}\left[w \circ X_t\right]\}$. Setting $y_t := \mathbf{E}\left[w \circ X_t\right]$, this becomes $y_{t+1} \leq \kappa(y_t)$. Using the fact that κ is increasing, we can then iterate backwards to obtain

$$\mathbf{E}\left[w \circ X_t\right] = y_t \leq \kappa^t(y_0).$$

Since $y_0 = \mathbf{E}[w \circ X_0] = w(x_0)$, the proof is now done. □

Assuming that the process (1) is ergodic and κ^t converges, we can also obtain a bound for the stationary distribution of the process.

Proposition 3.2. *If, in addition to the conditions of proposition 3.1,*

(a) *the process* (1) *is ergodic with stationary distribution* ψ^**,*
(b) w *is continuous, and*
(c) $\kappa^t[w(x_0)] \to M$ *as* $t \to \infty$*, then*

$$\int w d\psi^* \leq M.$$

Proof. Assume the conditions of the proposition. Ergodicity implies convergence of ψ_t to ψ^* in total variation. In turn, total variation convergence implies that, for every bounded measurable $h \colon S \to \mathbb{R}$, we have $\int h d\psi_t \to \int h d\psi^*$. So let s_n be the indicator function of the closed ball of radius n, and let $h_n := s_n \cdot w$. Since w is continuous, and therefore bounded on compact sets, it follows that h_n is bounded on S. Moreover, $h_n \uparrow w$ pointwise on S. Therefore,

$$\begin{aligned}\int w \, d\psi^* &= \lim_n \int h_n \, d\psi^* \qquad (\because \text{ Monotone Convergence Theorem}) \\ &= \lim_n \lim_t \int h_n \, d\psi_t \quad (\because \ h_n \text{ is bounded and measurable}) \\ &\leq \lim_n \lim_t \int w \, d\psi_t \leq \lim_n \lim_t \kappa^t [w(x_0)] = M.\end{aligned}$$

□

4. Additive Shock Models

We now specialize (1) to the common case where the shock ξ_t is additive. Precisely, we assume that the state space S is equal to $\mathbb{R}^n$, that ξ_t also takes values in S, and that $h(x, z) = g(x) + z$, where $g \colon S \to S$ is a measurable function. Thus,

$$X_{t+1} = g(X_t) + \xi_{t+1}, \quad X_0 = x_0 \in S, \quad \{\xi_t\}_{t=0}^{\infty} \overset{\text{IID}}{\sim} \varphi. \tag{3}$$

Let $B_r := \{x \in S : \|x\| \leq r\}$, and let $\{X_t\}_{t=0}^{\infty}$ be the sequence defined by (3). As before, let ψ_t be the distribution of X_t. Applying proposition 3.1, we now show that, under a growth condition on g and an exponential bound on the tails of ξ_t, the marginal distributions of the process $\{X_t\}_{t=0}^{\infty}$ have exponentially decreasing tails.

Proposition 4.1. *If*

$$\exists \, c \in \mathbb{R}_+ \textit{ and } \gamma \in (0,1) \textit{ such that, } \forall \, x \in S, \ \|g(x)\| \leq c + \gamma \|x\|, \tag{4}$$

then, for all $t \in \mathbb{N}$ and all $r > 0$, we have

$$\psi_t \left(S \setminus B_r\right) = \mathbf{P}\{\|X_t\| > r\} \leq \left[e^c \int e^{\|z\|} \varphi(dz) \right]^{\frac{1}{1-\gamma}} e^{\gamma^t \|x_0\| - r}. \tag{5}$$

The growth condition (4) permits g to be discontinuous and highly nonlinear. It is equivalent to the statement that there exists a hypersphere $B \subset S = \mathbb{R}^n$ centered on the origin such that $\|g(x)\|$ is bounded for $x \in B$, and on the complement of B the map g is contracting, in the sense that $\exists \gamma \in (0,1)$ such that $\|g(x)\| \leq \gamma \|x\|$ for all $x \in S \setminus B$. Similar restrictions have been used elsewhere in economic modeling. See, for example, Duffie and Singleton (1993).

Proof of Proposition 4.1. If $\int e^{\|z\|}\varphi(dz) = \infty$ then the bound is trivial, so suppose instead that this term is finite. We claim that Condition 3.1 is satisfied for $w(x) := e^{\|x\|}$ and

$$\kappa(s) := \beta s^{\gamma}, \quad \text{where } \beta := e^{c} \int e^{\|z\|}\varphi(dz).$$

(Since $\gamma \in (0,1)$, this function is concave and increasing.) To verify the claim, we must prove that

$$\int \exp(\|g(x) + z\|)\varphi(dz) \leq \kappa(e^{\|x\|}) = \beta e^{\gamma\|x\|}.$$

By the growth condition (4) we have

$$\|g(x) + z\| \leq \|g(x)\| + \|z\| \leq c + \gamma\|x\| + \|z\|.$$

$$\therefore \quad \int \exp(\|g(x) + z\|)\varphi(dz) \leq e^{c} \int e^{\|z\|}\varphi(dz) e^{\gamma\|x\|} = \beta e^{\gamma\|x\|}.$$

$$\therefore \quad \int w[g(x) + z]\varphi(dz) \leq \kappa[w(x)].$$

Apply Proposition 3.1 now yields

$$\mathbf{E}\, e^{\|X_t\|} = \mathbf{E}\, w(X_t) \leq \kappa^{t}[w(x_0)] = \beta^{\sum_{i=0}^{t}\gamma^{i}}[w(x_0)]^{\gamma^{t}}. \tag{6}$$

Since $w(x_0) = e^{\|x_0\|}$ and $\beta^{\sum_{i=0}^{t}\gamma^{i}} \leq \beta^{1/(1-\gamma)}$, we then have

$$\mathbf{E}\, e^{\|X_t\|} \leq \beta^{1/(1-\gamma)} e^{\gamma^{t}\|x_0\|} = \left[e^{c} \int e^{\|z\|}\varphi(dz)\right]^{\frac{1}{1-\gamma}} e^{\gamma^{t}\|x_0\|}.$$

The bound (5) now follows from (2). □

Condition (4) also has stability implications. In particular, if the condition holds and the shock process is sufficiently mixing, then global stability obtains. These kinds of results are well-known, and the next result provides details. (A full proof is given in the appendix.)

Theorem 4.1. *Let* $\{X_t\}_{t=0}^{\infty}$ *be the sequence defined by* (3)*. If* (4) *holds,* $\mathbf{E}\|\xi_t\| < \infty$ *and, in addition, the distribution* φ *admits a density representation that is continuous and strictly positive on* S*, then* $\{X_t\}_{t=0}^{\infty}$ *is geometrically ergodic.*

Under the conditions of theorem 4.1, the stationary distribution ψ^* of the state variable (and the long-run equilibrium of the system) inherits a tail bound similar to (5).

Proposition 4.2. *Let $\{X_t\}_{t=0}^{\infty}$ be the sequence defined by* (3)*. If the conditions of theorem 4.1 hold, then ψ^* satisfies*

$$\psi^* (S \setminus B_r) \leq \left[e^c \int e^{\|z\|} \varphi(dz) \right]^{\frac{1}{1-\gamma}} e^{-r} \qquad (r \geq 0). \tag{7}$$

As above, we are using the notation $B_r := \{x \in S : \|x\| \leq r\}$. Note that, in contrast to (5), this bound does not depend on x_0.

Proof of Proposition 4.2. The proof can be obtained from proposition 3.2, but in this case the result also follows directly from (5). If $\int e^{\|z\|} \varphi(dz) = \infty$ then the bound is trivial, so let us suppose that this term is finite. Fix $r \geq 0$. Using ergodicity and (5), we have

$$\psi^* (S \setminus B_r) = \lim_{t \to \infty} \psi_t (S \setminus B_r) \leq \left[e^c \int e^{\|z\|} \varphi(dz) \right]^{\frac{1}{1-\gamma}} \lim_{t \to \infty} e^{\gamma^t \|x_0\| - r}.$$

Since $\gamma \in (0, 1)$, the proof of (7) is done. □

5. Application

As an example, consider the self-exciting threshold autoregression model, which has found many applications in macroeconomic modeling.[5] It has the form

$$X_{t+1} = \sum_{k=1}^{K} (A_k X_t + b_k) \mathbb{1}\{X_t \in B_k\} + \xi_{t+1}, \tag{8}$$

where $(B_k)_{k=1}^K \subset \mathscr{B}(S)$ is a partition of $S = \mathbb{R}^n$, each A_k is an $n \times n$ matrix, and each b_k is an $n \times 1$ vector. The structure of the model is such that when the state is in the region B_k, the state variable follows the regime $x \mapsto A_k x + b_k$. This structure allows for significant nonlinearities.

Without any loss of generality, suppose that the first $1, \ldots, J$ elements of the partition $(B_k)_{k=1}^K$ are unbounded, and the remaining $J + 1, \ldots, K$ are bounded. Let B be the union of the bounded elements $B_{J+1}, \ldots, B_K$. Evidently g is bounded on bounded sets, so

[5] See, for example, Hansen (2001), or Taylor (2001).

$a := \sup_{x \in B} \|g(x)\|$ is finite. Finally, set $b := \sup_{1 \leq k \leq J} \|b_k\|$, and $\rho := \max_{1 \leq k \leq J} \rho_k$, where ρ_k is the spectral radius of A_k.

Proposition 5.1. *Let $\{X_t\}_{t=0}^{\infty}$ be defined by* (8)*, with $X_0 = x_0 \in S$ given. If $\rho < 1$, and if the distribution of ξ_t is multivariate normal, then $\{X_t\}_{t=0}^{\infty}$ is geometrically ergodic, and the tail bounds* (5) *and* (7) *hold when $c := a + b$ and $\gamma := \rho$.*

Proof. We need to show that (4) holds for

$$c = a + b, \quad \gamma = \rho, \quad \text{and} \quad g(x) = \sum_{k=1}^{K} (A_k x + b_k) \mathbb{1}_{B_k}(x).$$

For $x \notin B$ we have

$$\begin{aligned} \|g(x)\| &= \left\| \sum_{k=1}^{J} (A_k x + b_k) \mathbb{1}_{B_k}(x) \right\| \\ &\leq \sup_{1 \leq k \leq J} \|A_k x + b_k\| \leq \sup_{1 \leq k \leq J} \|A_k x\| + \sup_{1 \leq k \leq J} \|b_k\| \leq \rho \|x\| + b. \end{aligned}$$

On the other hand, for $x \in B$ we have $\|g(x)\| \leq a$ by definition. As a result, whether $x \in B$ or $x \in S \setminus B$ we have

$$\|g(x)\| \leq a + \rho\|x\| + b = c + \gamma\|x\|.$$

This confirms that (4) holds with $c := a + b$ and $\gamma := \rho$. Moreover, since the distribution of ξ_t is Gaussian, the conditions of theorem 4.1 are clearly satisfied. This implies both geometric ergodicity and the tail bounds (5) and (7). $\square$

To illustrate this result, consider Taylor's (2001) study of exchange rate dynamics and purchasing power parity (PPP). He uses a threshold autoregression of the form

$$X_{t+1} = \begin{cases} -\theta + \pi(X_t + \theta) + \xi_{t+1}, & \text{if } X_t < -\theta; \\ X_t + \xi_{t+1}, & \text{if } -\theta \leq X_t \leq \theta; \\ \theta + \pi(X_t - \theta) + \xi_{t+1}, & \text{if } X_t > \theta. \end{cases} \tag{9}$$

Here X represents the proportional deviation of the real exchange rate from PPP. The idea of the model is that trade frictions result in a "band of inaction," given here by $[-\theta, \theta]$. In this band, transaction costs imply that no arbitrage is possible. Outside $[-\theta, \theta]$ there is drift back towards the band, assuming that $\pi \in [0, 1)$. The shock sequence $\{\xi_t\}$ is taken to be IID and $N(0, \sigma^2)$.

Using the notation preceding Proposition 5.1, we can set $B = [-\theta, \theta]$, whence $a = \sup_{x \in B} |g(x)| = \theta$, and

$$b = \sup\{|(1 - \pi)\theta|, |(-\pi + 1)\theta|\} = (1 - \pi)\theta,$$

so that $c = a + b = (2 - \pi)\theta$. Also, ρ is the slope coefficient π. Applying these constants to Proposition 5.1 gives the equilibrium bound

$$\psi^* (S \setminus B_r) \leq \left[e^{(2-\pi)\theta} \int e^{\|z\|} \varphi(dz) \right]^{\frac{1}{1-\pi}} e^{-r}, \tag{10}$$

where ψ^* is the stationary distribution associated with (9).

6. Appendix

Proof of Theorem 4.1. Combining Theorem 15.0.1 and Lemma 15.2.8 in Meyn and Tweedie (2009), the Markov chain $\{X_t\}_{t=0}^{\infty}$ generated on S by (3) is geometrically ergodic whenever it is irreducible, aperiodic, precompact sets are petite, and there exists a coercive function $V : \mathbb{R}^n \to \mathbb{R}_+$ and positive constants L and λ such that $\lambda < 1$ and

$$\int V[g(x) + z] \varphi(dz) \leq \lambda V(x) + L \quad \text{for all } x \in \mathbb{R}^n. \tag{11}$$

(For definitions of irreducibility, aperiodicity, petite sets and coercive functions, see Meyn and Tweedie (2009, §4.2.1, §5.4.3, §5.5.2 and §9.4.1 respectively). A sufficient condition for a Markov chain to be irreducible and aperiodic is that any set $B \in \mathscr{B}(S)$ of positive Lebesgue measure can be reached in one step from any $x \in S$ with positive probability, which is to say that

$$\int \mathbb{1}\{g(x) + z \in B\} \varphi(z) dz = \int_{B - g(x)} \varphi(z) dz > 0.$$

This is immediate from the assumption that $\varphi > 0$ almost everywhere.

For a set $C \in \mathscr{B}(S)$ to be petite it is sufficient that there exists a measurable function $f\colon S \to [0,\infty)$ with $\int_S f > 0$ and

$$x \in C \quad \text{implies} \quad \varphi(y - g(x)) \geq f(y),\ \forall y \in S. \tag{12}$$

Let C be any bounded set, and let $\delta := \inf_{x,y \in C \times C} \varphi(y - g(x))$. If C has positive measure, and if $\delta > 0$, then we can take $f := \delta \mathbb{1}_C$, because if $x \in C$ then by the definition of δ we have $\varphi(y - g(x)) \geq f(y) = \delta \mathbb{1}_C(y)$.[6] But $\delta > 0$ must always hold for bounded C, because if C is bounded then it must be contained in some ball of size L, so that when $(x, y) \in C \times C$ we have

$$\|y - g(x)\| \leq \|y\| + \|g(x)\| \leq \|y\| + c + \gamma\|x\| \leq c + (1+\gamma)L =: M.$$

Thus $\delta = \inf_{x,y \in C \times C} \varphi(y - g(x)) \geq \inf_{\|z\| \leq M} \varphi(z)$, which is strictly positive because φ is strictly positive and continuous. We conclude that all bounded sets of positive measure are petite. Since subsets of petite sets are petite, it follows that all bounded sets are petite. Since the state space is $\mathbb{R}^n$, the bounded sets and the precompact sets are identical. We have now shown that all precompact sets are petite.

It remains only to show the existence of a coercive function $V\colon \mathbb{R}^n \to \mathbb{R}_+$ and positive constants L and λ such that $\lambda < 1$ and (11) holds. A function V is coercive if its sublevel sets are precompact. Since $x \mapsto \|x\|$ has this property we take $V(x) = \|x\|$. In view of (4), we have

$$\begin{aligned} \int \|g(x) + z\| \varphi(dz) &\leq \|g(x)\| + \int \|z\| \varphi(dz) \\ &\leq c + \gamma\|x\| + \int \|z\| \varphi(dz) = \lambda\|x\| + L, \end{aligned}$$

where $\lambda := \gamma$ and $L := c + \int \|z\| \varphi(dz)$. Since $\gamma < 1$ by assumption, the proof is now done. □

References

[1] Borovkov, A.A. (1998): *Ergodicity and Stability of Stochastic Processes*, Wiley, New York.

[6] Consider the two cases $y \in C$ and $y \notin C$.

[2] Duffie, D. and K. J. Singleton (1993): "Simulated Moments Estimation of Markov Models of Asset Prices," *Econometrica,* 61 (4) 929–952.

[3] Hansen, B. (2001): "Threshold Autoregression with Unit Root," *Econometrica,* 69 (6), 1555–96.

[4] Lux, T. (1998): "The Socio-Economic Dynamics of Speculative Markets: Interacting Agents, Chaos, and the Fat Tails of Return Distributions," *Journal of Economic Behavior and Organization,* 33, 2, 143–165.

[5] Mandelbrot, B.B. (1963): "The Variation of Certain Speculative Prices," *Journal of Business,* 36, 394–419.

[6] Meyn, S. and R. L. Tweedie (2009): *Markov Chains and Stochastic Stability,* 2nd Edition, Cambridge University Press, Cambridge.

[7] Pellicer-Lostao, C. and R. López-Ruiza (2010): "A chaotic gas-like model for trading markets," *Journal of Computational Science,* 1, 1, 24-32.

[8] Rachev, S.T. (ed., 2001): *Handbook of Heavy Tailed Distributions in Finance,* Elsevier, North-Holland.

[9] Stokey, N. L., R. E. Lucas and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics,* Harvard University Press, Massachusetts.

[10] Taylor, J.C. (1997): *An Introduction to Measure and Probability,* Springer-Verlag, New York.

[11] Taylor, A.M. (2001): "Potential Pitfalls for the Purchasing Power Parity Puzzle? Sampling and Specification Biases in Mean-Reversion Tests of the Law of One Price," *Econometrica,* 69 (2), 473–498.

RESEARCH SCHOOL OF ECONOMICS, AUSTRALIAN NATIONAL UNIVERSITY, ACT 0200, AUSTRALIA

E-mail address: `john.stachurski@anu.edu.au`