# Continuation Value Methods for Sequential Decisions: Optimality and Efficiency[1]

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December 13, 2017

ABSTRACT. This paper provides systematic analysis of a solution method for sequential decision problems originally due to Jovanovic (1982). We show that the operator employed by this method is semiconjugate to the Bellman operator associated with the corresponding dynamic program and has essentially equivalent dynamic properties. At the same time, for a broad class of sequential problems, the effective state space for this operator is strictly smaller. We characterize the difference in terms of model primitives and provide a range of examples.

*Keywords:* Continuation values, dynamic programming, sequential decisions

## 1. INTRODUCTION

Many decision making problems involve choosing when to act in the face of risk and uncertainty. Examples include deciding if or when to accept a job offer, exit or enter a market, default on a loan, bring a new product to market, exploit some new technology, or exercise an option (see, e.g., McCall (1970), Jovanovic (1982), Hopenhayn (1992), Dixit and Pindyck (1994), Ericson and Pakes (1995), Arellano (2008), Perla and Tonetti (2014), Fajgelbaum et al. (2017), and Schaal (2017)).

[1]We thank Boyan Jovanovic, Takashi Kamihigashi, Daisuke Oyama and Hiroyuki Ozaki for many insightful comments. Financial support from ARC Discovery Grant DP120100321 is gratefully acknowledged.

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Such problems can be solved using conventional dynamic programming methods based around the Bellman equation. There is, however, an alternative approach—introduced by Jovanovic (1982) in the context of industry dynamics—that focuses on continuation values. The idea involves calculating the continuation value directly, using an operator referred to below as the Jovanovic operator. This technique is now well-known to economists and routinely employed in a large variety of applications (see, e.g., Gomes et al. (2001), Ljungqvist and Sargent (2008), Lise (2013), Moscarini and Postel-Vinay (2013), Fajgelbaum et al. (2017), and Schaal (2017)).

Despite the existence of these two parallel and commonly used methods, their theoretical connections and relative efficiency have hitherto received no general investigation. One cost of this status quo is that studies using continuation value methods have been compelled to provide their own optimality analysis piecemeal in individual applications (see, e.g., Jovanovic (1982), Moscarini and Postel-Vinay (2013), or Fajgelbaum et al. (2017)). A second cost is that the most effective choice of method vis-a-vis a given application is often unknown ex-ante, and revealed only by experimentation in particular settings.

Here we begin a systematic analysis of the relationship between these two methods in a generic optimal stopping setting. As a first step, we show that the Bellman operator and the Jovanovic operator are semiconjugate, implying that any fixed point of one of the operators is a direct translation of a fixed point of the other. Iterative sequences generated by the operators are also simple translations. Second, we add topological structure to the generic setting and show that the Bellman operator and Jovanovic operator are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate.

The results stated above all elucidate the natural similarity between Bellman and Jovanovic operators. Despite these similarities, there do however remain important differences in terms of efficiency and analytical convenience. These differences concern the dimensionality of the effective state spaces associated with each operator. In particular, for an important class of problems, referred to below as continuation decomposable problems, the effective state space of the continuation value function is strictly lower than that of the value function. We characterize this class in terms of the structure of reward and state transition functions and provide a range of examples.

Lower dimensionality simplifies both theory and computation. As one illustration we study the time complexity of iteration with the Jovanovic and Bellman operators and quantify the difference analytically. We find large efficiency gains—typically measured in orders of magnitude—for the Jovanovic operator in the presence of continuation decomposability. These gains arise because in such settings continuation value based methods mitigate the curse of dimensionality, one of the primary stumbling blocks for dynamic programming (Rust (1997)).

To ensure sufficient generality for economic applications, we embed our optimality and symmetry arguments in (a) a space of potentially unbounded functions endowed with the weighted supremum norm distance, and (b) a space of integrable functions with divergence measured by $L_p$ norm. In particular, unbounded rewards are permitted provided that they do not cause continuation values to diverge. In doing so we draw on and extend work on dynamic programming with unbounded rewards found in several economic studies, including Boyd (1990), Rincón-Zapatero and Rodríguez-Palmero (2003), Martins-da Rocha and Vailakis (2010), Kamihigashi (2014) and Bäuerle and Jaśkiewicz (2018).

The paper is structured as follows. Section 2 outlines the problem. Section 3 explores the symmetric theoretical properties of the Bellman and Jovanovic operators in terms of fixed points and convergence. Section 4 discusses the asymmetries of their relative efficiency. Section 5 provides applications. Longer proofs are deferred to the appendix.

## 2. SET UP

This section presents a generic optimal stopping problem and the key operators and optimality concepts. As a first step, we introduce some mathematical techniques and notation used in this paper.

2.1. **Preliminaries.** Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. If $f$ and $g$ are functions, then $(f \vee g)(x) := f(x) \vee g(x)$. Given a Polish space $\mathsf{Z}$ and Borel sets $\mathscr{B}$, let $m\mathscr{B}$ be all $\mathscr{B}$-measurable functions from $\mathsf{Z}$ to $\mathbb{R}$.[2] Given $\kappa \colon \mathsf{Z} \to (0, \infty)$, the *$\kappa$-weighted supremum norm* of $f \colon \mathsf{Z} \to \mathbb{R}$ is

$$\|f\|_\kappa := \sup_{z \in \mathsf{Z}} \frac{|f(z)|}{\kappa(z)}.$$

If $\|f\|_\kappa < \infty$, we say that $f$ is *$\kappa$-bounded*. The symbol $b_\kappa \mathsf{Z}$ denotes all $\mathscr{B}$-measurable functions from $\mathsf{Z}$ to $\mathbb{R}$ that are $\kappa$-bounded.

Given a probability measure $\pi$ on $(\mathsf{Z}, \mathscr{B})$ and a constant $p \geq 1$, let

$$\|f\|_p := \left( \int |f|^p \, \mathrm{d}\pi \right)^{1/p}.$$

Let $L_p(\pi)$ be all (equivalence classes of ) functions $f \in m\mathscr{B}$ for which $\|f\|_p < \infty$.

Both $(b_\kappa \mathsf{Z}, \| \cdot \|_\kappa)$ and $(L_p(\pi), \| \cdot \|_p)$ form Banach spaces.

A *stochastic kernel* $P$ on $\mathsf{Z}$ is a map $P \colon \mathsf{Z} \times \mathscr{B} \to [0, 1]$ such that $z \mapsto P(z, B)$ is $\mathscr{B}$-measurable for each $B \in \mathscr{B}$ and $B \mapsto P(z, B)$ is a probability measure for each $z \in \mathsf{Z}$. For all $t \in \mathbb{N}$, $P^t(z, B) := \int P(z', B) P^{t-1}(z, \mathrm{d}z')$ is the probability of a state transition from $z$ to $B \in \mathscr{B}$ in $t$ steps, where $P^1(z, B) := P(z, B)$. A $\mathsf{Z}$-valued stochastic process $\{Z_t\}$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *$P$-Markov* if

$$\mathbb{P}\{Z_{t+1} \in B \,|\, \mathscr{F}_t\} = \mathbb{P}\{Z_{t+1} \in B \,|\, Z_t\} = P(Z_t, B)$$

$\mathbb{P}$-almost surely for all $t \in \mathbb{N}_0$ and all $B \in \mathscr{B}$. Here $\{\mathscr{F}_t\}$ is the natural filtration induced by $\{Z_t\}$. In what follows, $\mathbb{P}_z$ evaluates probabilities conditional on $Z_0 = z$ and $\mathbb{E}_z$ is the corresponding expectations operator.

[2] A topological space $\mathsf{Z}$ is called *Polish* if it is separable and completely metrizable (such as any $G_\delta$ subset of $\mathbb{R}^n$). We use the Polish assumption below mainly to avoid measurability concerns in arguments that condition on measure zero sets. All applied problems of which we are aware have Polish state spaces.

2.2. **Optimal Stopping.** Let $\mathsf{Z}$ be a Polish space with Borel sets $\mathscr{B}$. For the purposes of this paper, an optimal stopping problem is a tuple $(\beta, c, P, r)$ where

- $\beta \in (0,1)$ is discount factor,
- $c \in m\mathscr{B}$ is a *flow continuation reward* function,
- $P$ is a stochastic kernel on $(\mathsf{Z}, \mathscr{B})$, and
- $r \in m\mathscr{B}$ is a *terminal reward function*.

The interpretation is as follows: At time $t$ an agent observes $Z_t$, the current realization of a $\mathsf{Z}$-valued $P$-Markov process $\{Z_t\}_{t \geq 0}$, and chooses between stopping and continuing. Stopping generates terminal reward $r(Z_t)$ while continuing yields flow continuation reward $c(Z_t)$. If the agent continues, the time $t+1$ state $Z_{t+1}$ is observed and the process repeats. Future rewards are discounted at rate $\beta$.

An $\mathbb{N}_0$-valued random variable $\tau$ is called a (finite) *stopping time* if $\mathbb{P}\{\tau < \infty\} = 1$ and $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. Let $\mathscr{M}$ denote all such stopping times. The *value function* $v^*$ for $(\beta, c, P, r)$ is defined at $z \in \mathsf{Z}$ by

$$v^*(z) := \sup_{\tau \in \mathscr{M}} \mathbb{E}_z \left\{ \sum_{t=0}^{\tau-1} \beta^t c(Z_t) + \beta^\tau r(Z_\tau) \right\}. \tag{1}$$

A stopping time $\tau \in \mathscr{M}$ is called *optimal* if it attains the supremum in (1). Assume that the value function solves the Bellman equation[3]

$$v^*(z) = \max \left\{ r(z),\, c(z) + \beta \int v^*(z') P(z, dz') \right\}. \tag{2}$$

The corresponding Bellman operator is

$$Tv(z) = \max \left\{ r(z), c(z) + \beta \int v(z') P(z, \mathrm{d}z') \right\}. \tag{3}$$

The *continuation value function* associated with this problem is defined at $z \in \mathsf{Z}$ by

$$\psi^*(z) := c(z) + \beta \int v^*(z') P(z, dz'). \tag{4}$$

[3] A sufficient condition is that $\mathbb{E}_z \left( \sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ for all $z \in \mathsf{Z}$, as can be shown by applying theorem 1.11 (claim 1) of Peskir and Shiryaev (2006). Below we add assumptions under which this condition is guaranteed.

Using (2) and (4), we observe that $\psi^*$ satisfies

$$\psi^*(z) = c(z) + \beta \int \max\left\{r(z'), \psi^*(z')\right\} P(z, \mathrm{d}z'). \tag{5}$$

Analogous to the Bellman operator, the *continuation value operator* or *Jovanovic operator* $Q$ is constructed such that the continuation value function $\psi^*$ is a fixed point:

$$Q\psi(z) = c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z'). \tag{6}$$

## 3. Symmetries Between the Operators

In this section we show that Bellman and Jovanovic operators are semiconjugate and discuss the implications.[4] The semiconjugate relationship is most easily shown using operator-theoretic notation. To this end, let $Ph(z) := \int h(z')P(z, \mathrm{d}z')$ for all integrable $h \in m\mathscr{B}$ and observe that the Bellman operator $T$ can then be expressed as $T = RL$ where

$$R\psi := r \vee \psi \quad \text{and} \quad Lv := c + \beta P v. \tag{7}$$

(For any two operators we write the composition $A \circ B$ more simply as $AB$.)

3.1. **General Theory.** Let $\mathcal{V}$ be a subset of $m\mathscr{B}$ such that $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. The set $\mathcal{V}$ is understood as a set of candidate value functions. (Specific classes of functions are considered in the next section.) Let $\mathcal{C}$ be defined by

$$\mathcal{C} := L\mathcal{V} = \{\psi \in m\mathscr{B} \colon\ \psi = c + \beta P v \ \text{ for some } v \in \mathcal{V}\}. \tag{8}$$

By definition, $L$ is a surjective mapping from $\mathcal{V}$ onto $\mathcal{C}$. It is also true that $R$ maps $\mathcal{C}$ into $\mathcal{V}$. Indeed, if $\psi \in \mathcal{C}$, then there exists a $v \in \mathcal{V}$ such that $\psi = Lv$, and $R\psi = RLv = Tv$, which lies in $\mathcal{V}$ by assumption.

**Lemma 3.1.** *On $\mathcal{C}$, the operator $Q$ satisfies $Q = LR$, and $Q\mathcal{C} \subset \mathcal{C}$.*

*Proof.* The first claim is immediate from the definitions. The second follows from the claims just established (i.e., $R$ maps $\mathcal{C}$ to $\mathcal{V}$ and $L$ maps $\mathcal{V}$ to $\mathcal{C}$). $\square$

[4] Notably, the general theory developed in section 3.1 has no restriction on $\beta$ values.

The preceding discussion implies that $Q$ and $T$ are *semiconjugate*, in the sense that $LT = QL$ on $\mathcal{V}$ and $TR = RQ$ on $\mathcal{C}$. Indeed, since $T = RL$ and $Q = LR$, we have $LT = LRL = QL$ and $TR = RLR = RQ$ as claimed. This leads to the next result:

**Proposition 3.1.** *The following statements are true:*

(1) *If $v$ is a fixed point of $T$ in $\mathcal{V}$, then $Lv$ is a fixed point of $Q$ in $\mathcal{C}$.*
(2) *If $\psi$ is a fixed point of $Q$ in $\mathcal{C}$, then $R\psi$ is a fixed point of $T$ in $\mathcal{V}$.*

*Proof.* To prove the first claim, fix $v \in \mathcal{V}$. By the definition of $\mathcal{C}$, $Lv \in \mathcal{C}$. Moreover, since $v = Tv$, we have $QLv = LTv = Lv$. Hence, $Lv$ is a fixed point of $Q$ in $\mathcal{C}$. Regarding the second claim, fix $\psi \in \mathcal{C}$. Since $R$ maps $\mathcal{C}$ into $\mathcal{V}$ as shown above, $R\psi \in \mathcal{V}$. Since $\psi = Q\psi$, we have $TR\psi = RQ\psi = R\psi$. Hence, $R\psi$ is a fixed point of $T$ in $\mathcal{V}$. $\square$

The following result says that, at least on a theoretical level, iterating with either $T$ or $Q$ is essentially equivalent.

**Proposition 3.2.** *$T^{t+1} = RQ^tL$ on $\mathcal{V}$ and $Q^{t+1} = LT^tR$ on $\mathcal{C}$ for all $t \in \mathbb{N}_0$.*

*Proof.* That the claim holds when $t = 0$ has already been established. Now suppose the claim is true for arbitrary $t$. By the induction hypothesis we have $T^t = RQ^{t-1}L$ and $Q^t = LT^{t-1}R$. Since $Q$ and $T$ are semiconjugate as shown above, we have $T^{t+1} = TT^t = TRQ^{t-1}L = RQQ^{t-1}L = RQ^tL$ and $Q^{t+1} = QQ^t = QLT^{t-1}R = LTT^{t-1}R = LT^tR$. Hence, the claim holds by induction. $\square$

The theory above is based on the primitive assumption of a candidate value function space $\mathcal{V}$ with properties $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. Similar results can be established if we start with a generic candidate continuation value function space $\mathscr{C}$ that satisfies $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Appendix A gives details.

3.2. **Symmetry under Weighted Supremum Norm.** Next we impose a weighted supremum norm on the domain of $T$ and $Q$ in order to compare contractivity, optimality and related properties. The following assumption generalizes the standard weighted supremum norm assumption of Boyd (1990).

**Assumption 3.1.** There exist a $\mathscr{B}$-measurable function $g\colon \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$ and $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\int |r(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2, \tag{9}$$

$$\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4, \tag{10}$$

$$\text{and} \int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{11}$$

The interpretation is that both $\mathbb{E}_z |r(Z_n)|$ and $\mathbb{E}_z |c(Z_n)|$ are small relative to some function $g$ such that $\mathbb{E}_z g(Z_t)$ does not grow too fast.[5] Slow growth in $\mathbb{E}_z g(Z_t)$ is imposed by (11), which can be understood as a geometric drift condition (see, e.g., Meyn and Tweedie (2009), chapter 15). Note that if both $r$ and $c$ are bounded, then assumption 3.1 holds for $n := 0$, $g := \|r\| \vee \|c\|$, $m := 1$ and $d := 0$.

Assumption 3.1 reduces to that of Boyd (1990) if we set $n = 0$. Here we admit consideration of future transitions to enlarge the set of possible weight functions. The value of this generalization is illustrated in section 5.

**Theorem 3.1.** *If assumption 3.1 holds, then there exist positive constants $m'$ and $d'$ such that for $\ell, \kappa\colon \mathsf{Z} \to \mathbb{R}$ defined by*[6]

$$\ell(z) := m' \left( \sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d' \tag{12}$$

*and $\kappa(z) := \ell(z) + m'|r(z)|$, the following statements are true:*

*(1) $Q$ is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$, with unique fixed point $\psi^* \in b_\ell \mathsf{Z}$.*
*(2) $T$ is a contraction mapping on $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$, with unique fixed point $v^* \in b_\kappa \mathsf{Z}$.*

The next result shows that the convergence rates of $Q$ and $T$ are the same. In stating it, $L$ and $R$ are as defined in (7), while $\rho \in (0, 1)$ is the contraction coefficient of $T$ derived in theorem 3.1 (see (23) in appendix B for details).

[5] One can show that if assumption 3.1 holds for some $n$, then it must hold for all $n' > n$. Hence the values of $n$ in (9) and (10) can be chosen independently.

[6] If assumption 3.1 holds for $n = 0$, then $\ell(z) = g(z) + d'$ and $\kappa(z) = m'|r(z)| + g(z) + d'$. To guarantee that $\ell$ and $\kappa$ are real-valued, here and below, we assume that $\mathbb{E}_z |r(Z_t)|, \mathbb{E}_z |c(Z_t)| < \infty$ for $t = 1, \cdots, n-1$, which holds trivially in most applications of interest.

**Proposition 3.3.** *If assumption 3.1 holds, then $R(b_\ell \mathsf{Z}) \subset b_\kappa \mathsf{Z}$, $L(b_\kappa \mathsf{Z}) \subset b_\ell \mathsf{Z}$, and for all $t \in \mathbb{N}_0$, the following statements are true:*

*(1)* $\left\|Q^{t+1}\psi - \psi^*\right\|_\ell \leq \rho \left\|T^t R\psi - v^*\right\|_\kappa$ *for all* $\psi \in b_\ell \mathsf{Z}$.
*(2)* $\left\|T^{t+1}v - v^*\right\|_\kappa \leq \left\|Q^t L v - \psi^*\right\|_\ell$ *for all* $v \in b_\kappa \mathsf{Z}$.

Proposition 3.3 extends proposition 3.2, and their connections can be seen by letting $\mathcal{V} := b_\kappa \mathsf{Z}$. Notably, claim (1) implies that $Q$ converges as fast as $T$, even when its convergence is with respect to a larger norm (since $\ell \leq \kappa$).

The two operators are also symmetric in terms of continuity of fixed points:

**Assumption 3.2.** (1) The stochastic kernel $P$ is Feller; that is, $z \mapsto \int h(z')P(z, \mathrm{d}z')$ is continuous and bounded on $\mathsf{Z}$ whenever $h$ is. (2) $c$, $r$, $\ell$, $z \mapsto \int |r(z')|P(z, \mathrm{d}z')$, and $z \mapsto \int \ell(z')P(z, \mathrm{d}z')$ are continuous.[7]

**Proposition 3.4.** *If assumptions 3.1–3.2 hold, then $\psi^*$ and $v^*$ are continuous.*

3.3. **Symmetry in $L_p$.** The results of the preceding section for the most part carry over if we switch the underlying space to $L_p$. This section provides details.

**Assumption 3.3.** The state process $\{Z_t\}$ admits a stationary distribution $\pi$ and the reward functions $r, c$ are in $L_q(\pi)$ for some $q \geq 1$.

**Theorem 3.2.** *If assumption 3.3 holds, then for all $1 \leq p \leq q$, we have*[8]

*(1) $Q$ is a contraction mapping on $\left(L_p(\pi), \|\cdot\|_p\right)$ of modulus $\beta$, and the unique fixed point of $Q$ in $L_p(\pi)$ is $\psi^*$.*
*(2) $T$ is a contraction mapping on $\left(L_p(\pi), \|\cdot\|_p\right)$ of modulus $\beta$, and the unique fixed point of $T$ in $L_p(\pi)$ is $v^*$.*

The following result implies that $Q$ and $T$ have the same rate of convergence in terms of $L_p$-norm distance.

[7] A sufficient condition for assumption 3.2-(2) is: $g$ and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|, \mathbb{E}_z |c(Z_t)|$ are continuous for $t = 0, ..., n$ (with $n$ as defined in assumption 3.1).

[8] We typically omit phrases such as "with probability one" or "almost surely" in what follows. Uniqueness of fixed points is up to a $\pi$-null set.

**Proposition 3.5.** *If assumption 3.3 holds, then for all* $1 \leq p \leq q$*, both* $R$ *and* $L$ *map* $L_p(\pi)$ *into itself. Moreover, for all* $1 \leq p \leq q$ *and* $t \in \mathbb{N}_0$*, the following statements hold:*

(1) $\left\|Q^{t+1}\psi - \psi^*\right\|_p \leq \beta \left\|T^t R\psi - v^*\right\|_p$ *for all* $\psi \in L_p(\pi)$*.*
(2) $\left\|T^{t+1}v - v^*\right\|_p \leq \left\|Q^t Lv - \psi^*\right\|_p$ *for all* $v \in L_p(\pi)$*.*

Proposition 3.5 is an extension of proposition 3.2 in an $L_p$ space, and their connections can be seen by letting $\mathcal{V} := L_p(\pi)$.

## 4. Asymmetries Between the Operators

The preceding results show that $T$ and $Q$ exhibit dynamics that are essentially symmetric. Nevertheless, for a large class of economic models, the effective state space for $Q$ is lower dimensional than that of $T$. This section provides definitions and analysis, with examples deferred to section 5. Throughout, we write

$$\mathsf{Z} = \mathsf{X} \times \mathsf{Y} \quad \text{and} \quad Z_t = (X_t, Y_t)$$

where $\mathsf{X}$ is a Borel subset of $\mathbb{R}^\ell$ and $\mathsf{Y}$ is a Borel subset of $\mathbb{R}^n$.

### 4.1. Continuation Decomposability.

We call an optimal stopping problem $(\beta, c, P, r)$ *continuation decomposable* if $c$ and $P$ are such that

(a) $(X_{t+1}, Y_{t+1})$ and $X_t$ are independent given $Y_t$ and
(b) $c$ is a function of $Y_t$ but not $X_t$.

Condition (a) implies that $P(z, \mathrm{d}z')$ can be represented by the conditional distribution of $(x', y')$ given $y$, denoted below by $F_y(x', y')$. On an intuitive level, continuation decomposable problems are those where some state variables matter only for terminal rewards.

The significance of continuation decomposability is that, for such models, the Jovanovic operator can be written as

$$Q\psi(y) = c(y) + \beta \int \max\left\{r(x', y'), \psi(y')\right\} \mathrm{d}F_y(x', y'). \tag{13}$$

Thus, $Q$ acts on functions defined over $\mathsf{Y}$ alone. In contrast, assuming the original state space is minimal in the sense that all state variables impact on the value function, $T$ continues to act on functions defined over all of $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$. The set $\mathsf{Y}$ is $\ell$ dimensions lower than $\mathsf{Z}$.

4.2. **Complexity Analysis.** Lower dimensionality aids both analytical convenience and efficiency. Regarding the latter, one way to compare the efficiency of $Q$ and $T$ is to consider the time complexity of continuation value function iteration (CVI) and value function iteration (VFI). Both finite and infinite space approximations are considered.

4.2.1. *Finite Space.* Let $\mathsf{X} = \times_{i=1}^{\ell} \mathsf{X}^i$ and $\mathsf{Y} = \times_{j=1}^{n} \mathsf{Y}^j$, where $X^i$ and $Y^j$ are subsets of $\mathbb{R}$. Each $X^i$ (resp., $Y^j$) is represented by a grid of $K_i$ (resp., $M_j$) points. Integration operations in both VFI and CVI are replaced by summations. We use $\hat{P}$ and $\hat{F}$ to denote the transition matrices (i.e., discretized stochastic kernels) for VFI and CVI respectively.[9]

Let $K := \Pi_{i=1}^{\ell} K_i$ and $M := \Pi_{j=1}^{n} M_j$ with $K = 1$ for $\ell = 0$. Let $n > 0$. There are $KM$ grid points on $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$ and $M$ grid points on $\mathsf{Y}$. The matrix $\hat{P}$ is $(KM) \times (KM)$ and $\hat{F}$ is $M \times (KM)$. VFI and CVI are implemented by the operators $\hat{T}$ and $\hat{Q}$ defined respectively by

$$\hat{T}\vec{v} := \vec{r} \vee (\vec{c} + \beta \hat{P} \vec{v}) \quad \text{and} \quad \hat{Q}\vec{\psi}_y := \vec{c}_y + \beta \hat{F}(\vec{r} \vee \vec{\psi}).$$

Here $\vec{q}$ represents a column vector with $i$-th element equal to $q(x_i, y_i)$, where $(x_i, y_i)$ is the $i$-th element of the list of grid points on $\mathsf{X} \times \mathsf{Y}$. Let $\vec{q}_y$ denote the column vector with the $j$-th element equal to $q(y_j)$, where $y_j$ is the $j$-th element of the list of grid points on $\mathsf{Y}$. The vectors $\vec{v}$, $\vec{r}$, $\vec{c}$ and $\vec{\psi}$ are $(KM) \times 1$, while $\vec{c}_y$ and $\vec{\psi}_y$ are $M \times 1$.

4.2.2. *Infinite Space.* We use the same number of grid points as before, but now for continuous state function approximation rather than discretization. In particular,

[9] See Tauchen and Hussey (1991) for a discussion of potential discretization methods.

we replace the discrete state summation with Monte Carlo integration. Assume that the transition function of the state process follows

$$X_{t+1} = f_1(Y_t, W_{t+1}), \quad Y_{t+1} = f_2(Y_t, W_{t+1}), \quad \{W_t\} \overset{\text{IID}}{\sim} \Phi.$$

After drawing $U_1, \cdots, U_N \overset{\text{IID}}{\sim} \Phi$, with $N$ being the MC sample size, CVI and VFI are implemented by

$$\hat{Q}\psi(y) := c(y) + \beta \frac{1}{N} \sum_{i=1}^{N} \max \left\{ r\left(f_1(y, U_i), f_2(y, U_i)\right), h\langle\psi\rangle \left(f_2(y, U_i)\right) \right\}$$

$$\text{and} \quad \hat{T}v(x, y) := \max \left\{ r(x, y), \, c(y) + \beta \frac{1}{N} \sum_{i=1}^{N} g\langle v \rangle \left(f_1(y, U_i), f_2(y, U_i)\right) \right\}.$$

Here $\psi = \{\psi(y)\}$, with $y$ in the set of grid points on $\mathsf{Y}$, and $v = \{v(x, y)\}$, with $(x, y)$ in the set of grid points on $\mathsf{X} \times \mathsf{Y}$. Moreover, $h\langle\cdot\rangle$ and $g\langle\cdot\rangle$ are interpolating functions for CVI and VFI respectively. For example, $h\langle\psi\rangle(z)$ can be understood as interpolating the vector $\psi$ to obtain a function $h\langle\psi\rangle$ and then evaluating at $z$.

4.2.3. *Time Complexity.* Table 1 provides the time complexity of CVI and VFI, estimated by counting the number of floating point operations. Each such operation is assumed to have complexity $\mathcal{O}(1)$.[10] Function evaluations associated with the model primitives are also assumed to be of order $\mathcal{O}(1)$.

TABLE 1. Time complexity: VFI v.s CVI

| Cmplx. | VFI: 1-loop | CVI: 1-loop | VFI: $n$-loop | CVI: $n$-loop |
|---|---|---|---|---|
| FS | $\mathcal{O}(K^2M^2)$ | $\mathcal{O}(KM^2)$ | $\mathcal{O}(nK^2M^2)$ | $\mathcal{O}(nKM^2)$ |
| IS | $\mathcal{O}(NKM\log(KM))$ | $\mathcal{O}(NM\log(M))$ | $\mathcal{O}(nNKM\log(KM))$ | $\mathcal{O}(nNM\log(M))$ |

Note: For IS approximation, binary search is used when we evaluate the interpolating function at a given point. The results hold for linear, quadratic, cubic, and $k$-nearest neighbors interpolations.

For both finite space (FS) and infinite space (IS) approximations, CVI provides better performance than VFI. For FS, CVI is more efficient than VFI by order $\mathcal{O}(K)$, while for IS, CVI is more efficient than VFI by order $\mathcal{O}\left(K\log(KM)/\log(M)\right)$. For

[10] Floating point operations are any elementary actions (e.g., $+$, $\times$, $\vee$, $\wedge$) on or assignments with floating point numbers. If $f$ and $g$ are scalar functions on $\mathbb{R}^n$, we write $f(x) = \mathcal{O}(g(x))$ whenever there exist $C, M > 0$ such that $\|x\| \geq M$ implies $|f(x)| \leq C|g(x)|$, where $\|\cdot\|$ is the sup norm.

example, if we have 250 grid points in each dimension, then in the FS case, evaluating a given number of loops will take around $250^{\ell}$ times longer via CVI than via VFI, after adjusting for order approximations.

See appendix C for a proof of the results in table 1.

## 5. Applications

We consider six applications. For the first two we discuss both optimality and continuation decomposability. For the remaining cases we discuss only the latter.

5.1. **Job Search.** Consider a worker who receives current wage offer $w_t$ and chooses to either accept and work permanently at that wage, or reject the offer, receive unemployment compensation $c_0$ and reconsider next period (see, e.g., McCall (1970) or Pissarides (2000)). The wage process $\{w_t\}_{t \geq 0}$ is assumed to be

$$w_t = \eta_t + \theta_t \xi_t, \quad \text{where} \quad \ln \theta_t = \rho \ln \theta_{t-1} + \ln \varepsilon_t \tag{14}$$

and $\{\xi_t\}$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ are positive IID innovations that are mutually independent. We interpret $\theta_t$ as the persistent component of labor income and allow it to be nonstationary. When $\eta_t$ is constant it can be interpreted as social security.[11] Viewed as an optimal stopping problem,

- the state is $z = (w, \theta)$, with stochastic kernel $P$ defined by (14),
- the terminal reward is $r(w) = u(w)/(1 - \beta)$, where $u$ is a utility function,
- and the flow continuation reward $c$ is the constant $u(c_0)$.

The model is continuation decomposable, as can be seen by letting $X_t := w_t$ and $Y_t := \theta_t$. In particular, $c$ does not depend on $w_t$ and $(w_{t+1}, \theta_{t+1})$ is independent of $w_t$ given $\theta_t$. Hence the effective state space for $Q$ is one-dimensional while that of

[11] Similar dynamics appear in many labor market, search-theoretic and real options studies (see e.g., Gomes et al. (2001), Low et al. (2010), Chatterjee and Eyigungor (2012), Bagger et al. (2014), Kellogg (2014)).

$T$ is two-dimensional. Letting $F_\theta(w', \theta')$ be the distribution of $(w_{t+1}, \theta_{t+1})$ given $\theta_t$, the Bellman operator satisfies

$$Tv(w, \theta) = \max \left\{ \frac{u(w)}{1-\beta}, \; u(c_0) + \beta \int v(w', \theta') \, \mathrm{d}F_\theta(w', \theta') \right\},$$

while the Jovanovic operator is

$$Q\psi(\theta) = u(c_0) + \beta \int \max \left\{ \frac{u(w')}{1-\beta}, \; \psi(\theta') \right\} \mathrm{d}F_\theta(w', \theta').$$

Whether or not assumptions 3.1–3.3 hold depends on the primitives. Suppose for example that

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \text{with} \quad u(w) = \ln w \quad \text{when } \gamma = 1. \tag{15}$$

We focus here on the case $\gamma = 1$ and $0 \leq \rho < 1$, although other cases such as $\gamma > 1$ and $-1 < \rho < 0$ can be treated with similar arguments.[12] We take $\varepsilon_t \sim LN(0, \sigma^2)$. Regarding assumptions 3.1–3.2, we assume that $\{\eta_t\}$, $\{\eta_t^{-1}\}$ and $\{\xi_t\}$ have finite first moments.

The reward function for this dynamic program is unbounded above and below, and the state space is likewise unbounded. Nevertheless, we can establish the key optimality results from section 3 as follows. First, choose $n \in \mathbb{N}_0$ such that $\beta \exp(\rho^{2n} \sigma^2) < 1$, and let

$$g(z) = g(w, \theta) = \theta^{\rho^n}.$$

To verify (9), we make use of the following technical lemma, which is obtained from the law of motion (14), and provides a bound on expected time $n$ wages in terms of initial condition $\theta_0 = \theta$. The proof is in appendix B.

**Lemma 5.1.** *For all $n \in \mathbb{N}_0$, (a) there exist a pair $A_n, B \in \mathbb{R}$ such that $\mathbb{E}_\theta |\ln w_n| \leq A_n \theta^{\rho^n} + B$, and (b) $\theta \mapsto \mathbb{E}_\theta |\ln w_n|$ is continuous.*

[12] One can also treat the nonstationary case $\rho = \pm 1$ under some further parametric assumptions using the weighted supremum norm techniques developed above. Details are available from the authors on request.

Now (9) can be established, since, conditioning on $\theta_0 = \theta$,

$$\mathbb{E}_{\theta}|r(w_n)| = \frac{\mathbb{E}_{\theta}|\ln w_n|}{1-\beta} \leq \frac{A_n}{1-\beta}\theta^{\rho^n} + \frac{B}{1-\beta} = \frac{A_n}{1-\beta} g(w,\theta) + \frac{B}{1-\beta}.$$

Condition (10) is trivial because $c$ is constant. To see that condition (11) holds, note that $\rho \in [0,1)$, so, conditioning on $\theta_0 = \theta$ once more,

$$\mathbb{E}_{\theta}\, g(w_1,\theta_1) = \mathbb{E}\, (\theta^{\rho}\varepsilon_1)^{\rho^n} = \theta^{\rho^{n+1}} \exp(\rho^{2n}\sigma^2/2) \leq (\theta^{\rho^n}+1)\exp(\rho^{2n}\sigma^2).$$

Hence (11) holds with $m = d = \exp(\rho^{2n}\sigma^2)$. Assumption 3.1 has now been established. By theorem 3.1 and proposition 3.3, $Q$ and $T$ are contraction mappings with the same rate of convergence. The above analysis also implies that assumption 3.2 holds (see footnote 7), so proposition 3.4 implies that both $v^*$ and $\psi^*$ are continuous.

We can also embed this problem in $L_p(\pi)$. To verify assumption 3.3, we assume that the distributions of $\eta_t$ and $\xi_t$ are represented respectively by densities $\mu$ and $\nu$, and that $\eta_t$, $\eta_t^{-1}$ and $\xi_t$ have finite $q$-th moments.

Since $\rho \in [0,1)$, the state process $\{(w_t,\theta_t)\}$ has stationary density

$$\pi(w,\theta) = f^*(\theta)\, p(w|\theta),$$

where $f^*(\theta) = LN(0, \sigma^2/(1-\rho^2))$ and $\int_A p(w|\theta)\,\mathrm{d}w = \int_{\{\eta+\theta\xi\in A\}} \mu(\eta)\nu(\xi)\,\mathrm{d}(\eta,\xi)$. Then the next lemma (proved in appendix B) implies that assumption 3.3 holds.

**Lemma 5.2.** *The reward functions $r$ and $c$ are in $L_q(\pi)$.*

By theorem 3.2 and proposition 3.5, $Q$ and $T$ are both contraction mappings with the same rate of convergence (in $L_p$ norm distances, for all $1 \leq p \leq q$).

5.2. **Search with Learning.** Consider a job search problem with learning (see, e.g., McCall (1970), Pries and Rogerson (2005), Nagypál (2007), or Ljungqvist and Sargent (2012)). The setup is as in section 5.1, except that $\{w_t\}_{t\geq 0}$ follows

$$\ln w_t = \xi + \varepsilon_t, \quad \text{where } \{\varepsilon_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0,\delta_\varepsilon).$$

Here $\xi$ is an unobservable mean over which the worker has prior $\xi \sim N(\mu,\delta)$. The worker's current estimate of the next period wage distribution is $f(w'|\mu,\delta) =$

$LN(\mu, \delta + \delta_\varepsilon)$. If the current offer is turned down, the worker updates his belief after observing $w'$. By Bayes' rule, the posterior satisfies $\xi | w' \sim N(\mu', \delta')$, where $\delta' = \nu(\delta) := 1/\left(1/\delta + 1/\delta_\varepsilon\right)$ and $\mu' = \phi(\mu, \delta, w') := \delta' \left(\mu/\delta + \ln w'/\delta_\varepsilon\right)$. Viewed as an optimal stopping problem,

- the state is $z = (w, \mu, \delta)$, and for each map $h$, the stochastic kernel $P$ satisfies
$$\int h(z')P(z, \mathrm{d}z') = \int h\left(w', \phi(\mu, \delta, w'), \nu(\delta)\right) f(w'|\mu, \delta)\, \mathrm{d}w'$$
- the reward functions are $r(w) = u(w)/(1-\beta)$ and $c \equiv u(c_0)$.

The model is continuation decomposable with $X_t := w_t$ and $Y_t := (\mu_t, \delta_t)$, since $r$ does not depend on $(\mu_t, \delta_t)$ and the next period state $(w_{t+1}, \mu_{t+1}, \delta_{t+1})$ is independent of $w_t$ once $\mu_t$ and $\delta_t$ are known. Letting $F_{\mu,\delta}(w', \mu', \delta')$ be the distribution of $(w_{t+1}, \mu_{t+1}, \delta_{t+1})$ given $(\mu_t, \delta_t)$, the Bellman and Jovanovic operators are, respectively,

$$Tv(w, \mu, \delta) = \max\left\{\frac{u(w)}{1-\beta},\ u(c_0) + \beta \int v(w', \mu', \delta')\, \mathrm{d}F_{\mu,\delta}(w', \mu', \delta')\right\}$$

$$\text{and} \quad Q\psi(\mu, \delta) = u(c_0) + \beta \int \max\left\{\frac{u(w')}{1-\beta},\ \psi(\mu', \delta')\right\} \mathrm{d}F_{\mu,\delta}(w', \mu', \delta').$$

Again, the domain of the candidate function space is one dimension lower for $Q$ than $T$.

Regarding optimality, suppose, for example, that the CRRA parameter $\gamma$ is greater than 1. (The case $\gamma = 1$ can be treated along similar lines.) Let $n = 1$ and let

$$g(w, \mu, \delta) = \mathrm{e}^{(1-\gamma)\mu + (1-\gamma)^2\delta/2}.$$

Condition (9) holds, since, conditioning on $(\mu_0, \delta_0) = (\mu, \delta)$,

$$\mathbb{E}_{\mu,\delta} |r(w_1)| = \frac{\mathbb{E}_{\mu,\delta}\, w_1^{1-\gamma}}{1-\beta} = \frac{\mathrm{e}^{(1-\gamma)^2\delta_\varepsilon/2}}{1-\beta}\, g(w, \mu, \delta).$$

Condition (10) is trivial since $c$ is constant. Condition (11) holds, since, conditioning on $(\mu_0, \delta_0) = (\mu, \delta)$, the expressions of $\mu'$ and $\delta'$ imply that

$$\mathbb{E}_{\mu,\delta}\, g(w_1, \mu_1, \delta_1) = \mathrm{e}^{(1-\gamma)^2\delta_1/2 + (1-\gamma)\delta_1\mu/\delta}\, \mathbb{E}_{\mu,\delta}\, w_1^{(1-\gamma)\delta_1/\delta_\varepsilon} = g(w, \mu, \delta).$$

Hence assumption 3.1 holds. Theorem 3.1 and proposition 3.3 imply that $Q$ and $T$ are contraction mappings with the same rate of convergence. The analysis above also implies that assumption 3.2 holds (see footnote 7), so $v^*$ and $\psi^*$ are continuous by proposition 3.4.

5.3. **Firm Entry.** Consider a condensed version of the firm entry problem in Fajgelbaum et al. (2017). At the start of period $t$, a firm observes a fixed cost $f_t$ and then decides whether to incur this cost and enter a market, earning stochastic payoff $\pi_t$, or wait and reconsider next period. The sequence $\{f_t\}$ is IID, while the current payoff $\pi_t$ is unknown prior to entry. The firm has prior belief $\phi(\pi; \theta_t)$, where $\phi$ is a distribution over payoffs that is parameterized by a vector $\theta_t$. If the firm does not enter then $\theta_t$ is updated via Bayesian learning. In an optimal stopping format,

- the state is $z = (f, \theta)$, with stochastic kernel $P$ defined by the distribution of $\{f_t\}$ and the Bayesian updating mechanism of $\{\theta_t\}$,
- the terminal reward is the entry payoff $r(f, \theta) = \int \pi \phi(\mathrm{d}\pi; \theta) - f$,
- and the flow continuation reward $c \equiv 0$.

This model is continuation decomposable, as can be seen by letting $X_t := f_t$ and $Y_t := \theta_t$. In particular, since $\{f_t\}$ is IID, $(f_{t+1}, \theta_{t+1})$ is independent of $f_t$ given $\theta_t$. Let $F_\theta(f', \theta')$ be the distribution of $(f_{t+1}, \theta_{t+1})$ given $\theta_t$. The Bellman operator is

$$Tv(f, \theta) = \max \left\{ \int \pi \phi(\mathrm{d}\pi; \theta) - f, \, \beta \int v(f', \theta') \, \mathrm{d}F_\theta(f', \theta') \right\},$$

while the Jovanovic operator is

$$Q\psi(\theta) = \beta \int \max \left\{ \int \pi \phi(\mathrm{d}\pi; \theta') - f', \, \psi(\theta') \right\} \mathrm{d}F_\theta(f', \theta').$$

5.4. **Research and Development.** Firm's R&D decisions are often modeled as a sequential search process for better technologies (see, e.g., Jovanovic and Rob (1989), Bental and Peled (1996), Perla and Tonetti (2014)). Each period, an idea of value $s_t$ is observed, and the firm decides whether to put this idea into productive use, or develop it further by investing in R&D. The former choice yields a payoff $r(s_t, k_t)$, where $k_t$ is the amount of capital input. The latter incurs a fixed cost $c_0 > 0$

(that renders $k_{t+1} = k_t - c_0$) and creates a new technology $s_{t+1}$ next period. Let $\{s_t\} \overset{\text{IID}}{\sim} \mu$. Viewed as an optimal stopping problem,

- the state is $z = (s, k)$, and for given map $h$, the stochastic kernel $P$ satisfies
$$\int h(z')P(z, \mathrm{d}z') = \int h\left(s', k - c_0\right) \mu(\mathrm{d}s'),$$
- the terminal reward is $r(s, k)$ and the flow continuation reward is $c \equiv -c_0$.

This model is also continuation decomposable, as can be seen by letting $X_t := s_t$ and $Y_t := k_t$. Let $F_k(s', k')$ be the distribution of $(s_{t+1}, k_{t+1})$ given $k_t$. The Bellman and Jovanovic operators are respectively

$$Tv(s, k) = \max\left\{r(s, k), -c_0 + \beta \int v(s', k')\, \mathrm{d}F_k(s', k')\right\}$$

$$\text{and} \quad Q\psi(s) = -c_0 + \beta \int \max\left\{r(s', k'), \psi(s')\right\} \mathrm{d}F_k(s', k').$$

5.5. **Real Options.** Consider a general financial/real option framework (see, e.g., Dixit and Pindyck (1994), Alvarez and Dixit (2014), and Kellogg (2014)). Let $p_t$ be the current price of a certain financial/real asset and $\lambda_t$ another state variable. The process $\{\lambda_t\}$ is $\Phi$-Markov and affects $\{p_t\}$ via $p_t = f(\lambda_t, \varepsilon_t)$, where $\{\varepsilon_t\} \overset{\text{IID}}{\sim} \mu$ and is independent of $\{\lambda_t\}$. Let $K$ be the strike price of the asset. Each period, the agent decides whether to exercise the option now (i.e., purchase the asset at price $K$), or wait and reconsider next period. In an optimal stopping format,

- the state is $z = (p, \lambda)$, and for given map $h$, the stochastic kernel $P$ satisfies
$$\int h(z')P(z, \mathrm{d}z') = \int h\left(f(\lambda', \varepsilon'),\, \lambda'\right) \mu(\mathrm{d}\varepsilon')\, \Phi(\lambda, \mathrm{d}\lambda'),$$
- the terminal reward (exercise the option now) is $r(p) = (p - K)^+$,
- and the flow continuation reward is $c \equiv 0$.

The model is continuation decomposable, with $X_t := p_t$ and $Y_t := \lambda_t$. Let $F_\lambda(p', \lambda')$ be the distribution of $(p_{t+1}, \lambda_{t+1})$ conditional on $\lambda_t$. The Bellman and Jovanovic operators are, respectively,

$$Tv(p, \lambda) = \max\left\{(p - K)^+, \beta \int v(p', \lambda')\, \mathrm{d}F_\lambda(p', \lambda')\right\}$$

$$\text{and} \quad Q\psi(\lambda) = \beta \int \max\left\{(p' - K)^+, \psi(\lambda')\right\} \mathrm{d}F_\lambda(p', \lambda').$$

5.6. **Transplants.** In health economics, a well-known problem concerns the decision of a surgeon to accept/reject a transplantable organ for the patient (see, e.g., Alagoz et al., 2004). The surgeon aims to maximize the reward of the patient. Each period, she receives an organ offer of quality $q_t$, where $\{q_t\} \overset{\text{IID}}{\sim} G$. The patient's health $h_t$ evolves according to a $H$-Markov process if the surgeon rejects the organ. If she accepts this organ for transplant, the operation succeeds with probability $p(q_t, h_t)$, and confers benefit $B(h_t)$ to the patient, while a failed operation results in death. The patient's single period utility when alive is $u(h_t)$. Viewed as an optimal stopping problem,

- the state is $z = (q, h)$, and for a given map $f$, the stochastic kernel $P$ satisfies
$$\int f(z')P(z, \mathrm{d}z') = \int f(q', h')G(\mathrm{d}q')H(h, \mathrm{d}h'),$$
- the terminal reward (accept the offer) is $r(q, h) = u(h) + p(q, h)B(h)$,
- and the flow continuation reward is $c(h) = u(h)$.

This model is continuation decomposable by letting $X_t := q_t$ and $Y_t := h_t$. Let $F_h(q', h')$ be the distribution of $(q_{t+1}, h_{t+1})$ given $h_t$. The Bellman and Jovanovic operators are respectively

$$Tv(q, h) = \max\left\{u(h) + p(q, h)B(h),\ u(h) + \beta \int v(q', h')\, \mathrm{d}F_h(q', h')\right\}$$

$$\text{and} \quad Q\psi(h) = u(h) + \beta \int \max\left\{u(h') + p(q', h')B(h'),\ \psi(h')\right\} \mathrm{d}F_h(q', h').$$

## Appendix A: Some Lemmas

To see the symmetric properties of $Q$ and $T$ from an alternative perspective, we start our analysis with a generic candidate continuation value function space. Let $\mathscr{C}$ be a subset of $m\mathscr{B}$ such that $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Let $\mathscr{V}$ be defined by

$$\mathscr{V} := R\mathscr{C} = \{v \in m\mathscr{B} \colon v = r \vee \psi \ \text{ for some } \psi \in \mathscr{C}\}. \tag{16}$$

Then $R$ is a surjective map from $\mathscr{C}$ onto $\mathscr{V}$, $Q = LR$ on $\mathscr{C}$ and $T = RL$ on $\mathscr{V}$. The following result parallels the theory of section 3.1, and is helpful for deriving important convergence properties once topological structure is added to the generic setting, as to be shown.

**Lemma 5.3.** *The following statements are true:*

*(1) $L\mathscr{V} \subset \mathscr{C}$ and $T\mathscr{V} \subset \mathscr{V}$.*
*(2) If $v$ is a fixed point of $T$ in $\mathscr{V}$, then $Lv$ is a fixed point of $Q$ in $\mathscr{C}$.*
*(3) If $\psi$ is a fixed point of $Q$ in $\mathscr{C}$, then $R\psi$ is a fixed point of $T$ in $\mathscr{V}$.*
*(4) $T^{t+1} = RQ^tL$ on $\mathscr{V}$ and $Q^{t+1} = LT^tR$ on $\mathscr{C}$ for all $t \in \mathbb{N}_0$.*

*Proof.* The proof is similar to that of propositions 3.1–3.2 and thus omitted. □

**Lemma 5.4.** *Under assumption 3.1, there exist $b_1, b_2 \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,*

*(1) $|v^*(z)| \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + b_1 g(z) + b_2$.*
*(2) $|\psi^*(z)| \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z|r(Z_t)| + \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z|c(Z_t)| + b_1 g(z) + b_2$.*

*Proof.* Without loss of generality, we assume $m \neq 1$ in assumption 3.1. By that assumption, $\mathbb{E}_z|r(Z_n)| \leq a_1 g(z) + a_2$, $\mathbb{E}_z|c(Z_n)| \leq a_3 g(z) + a_4$ and $\mathbb{E}_z g(Z_1) \leq m g(z) + d$ for all $z \in \mathsf{Z}$. For all $t \geq 1$, by the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3),

$$\mathbb{E}_z g(Z_t) = \mathbb{E}_z\left[\mathbb{E}_z\left(g(Z_t)|\mathscr{F}_{t-1}\right)\right] = \mathbb{E}_z\left(\mathbb{E}_{Z_{t-1}} g(Z_1)\right) \leq m\,\mathbb{E}_z g(Z_{t-1}) + d.$$

Induction shows that for all $t \geq 0$,

$$\mathbb{E}_z g(Z_t) \leq m^t g(z) + \frac{1-m^t}{1-m} d. \tag{17}$$

Moreover, for all $t \geq n$, applying the Markov property again yields

$$\mathbb{E}_z|r(Z_t)| = \mathbb{E}_z\left[\mathbb{E}_z\left(|r(Z_t)||\mathscr{F}_{t-n}\right)\right] = \mathbb{E}_z\left(\mathbb{E}_{Z_{t-n}}|r(Z_n)|\right) \leq a_1\,\mathbb{E}_z g(Z_{t-n}) + a_2.$$

By (17), for all $t \geq n$, we have

$$\mathbb{E}_z|r(Z_t)| \leq a_1\left(m^{t-n} g(z) + \frac{1-m^{t-n}}{1-m} d\right) + a_2. \tag{18}$$

Similarly, for all $t \geq n$, we have

$$\mathbb{E}_z|c(Z_t)| \leq a_3\,\mathbb{E}_z g(Z_{t-n}) + a_4 \leq a_3\left(m^{t-n} g(z) + \frac{1-m^{t-n}}{1-m} d\right) + a_4. \tag{19}$$

Let $S(z) := \sum_{t \geq 1} \beta^t \mathbb{E}_z [|r(Z_t)| + |c(Z_t)|]$. Based on (17)–(19), we can show that

$$S(z) \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + \frac{a_1 + a_3}{1 - \beta m} g(z) + \frac{(a_1 + a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}. \tag{20}$$

Since $|v^*| \leq |r| + |c| + S$ and $|\psi^*| \leq |c| + S$, the two claims hold by letting $b_1 := \frac{a_1+a_3}{1-\beta m}$ and $b_2 := \frac{(a_1+a_3)d+a_2+a_4}{(1-\beta m)(1-\beta)}$. □

**Lemma 5.5.** *Under assumption 3.1, the value function solves the Bellman equation*

$$v^*(z) = \max\left\{r(z),\, c(z) + \beta \int v^*(z')P(z, dz')\right\} = \max\left\{r(z),\, \psi^*(z)\right\}. \tag{21}$$

*Proof of lemma 5.5.* By theorem 1.11 of Peskir and Shiryaev (2006), it suffices to show that $\mathbb{E}_z \left(\sup_{k \geq 0} \left|\sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k)\right|\right) < \infty$ for all $z \in \mathsf{Z}$. This is true since

$$\sup_{k \geq 0} \left|\sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k)\right| \leq \sum_{t \geq 0} \beta^t [|r(Z_t)| + |c(Z_t)|] \tag{22}$$

with probability one, and by the monotone convergence theorem and lemma 5.4 (see (20) in appendix A), the right hand side of (22) is $\mathbb{P}_z$-integrable for all $z \in \mathsf{Z}$. □

## Appendix B : Main Proofs

*Proof of theorem 3.1.* Let $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$. Since $\beta m < 1$ by assumption 3.1, we can choose positive constants $m'$ and $d'$ such that

$$m + d_1 m' > 1, \quad \rho := \beta(m + d_1 m') < 1 \quad \text{and} \quad d' \geq (d_2 m' + d)/(m + d_1 m' - 1). \tag{23}$$

Regarding claim (1), we first show that $Q$ is a contraction mapping on $b_\ell \mathsf{Z}$ with modulus $\rho$. By the weighted contraction mapping theorem (see, e.g., Boyd (1990), section 3), it suffices to verify: (a) $Q$ is monotone, i.e., $Q\psi \leq Q\phi$ if $\psi, \phi \in b_\ell \mathsf{Z}$ and $\psi \leq \phi$; (b) $Q0 \in b_\ell \mathsf{Z}$ and $Q\psi$ is $\mathscr{B}$-measurable for all $\psi \in b_\ell \mathsf{Z}$; and (c) $Q(\psi + a\ell) \leq Q\psi + a\rho\ell$ for all $a \in \mathbb{R}_+$ and $\psi \in b_\ell \mathsf{Z}$. Obviously, condition (a) holds. By (6) and (12), we have

$$\frac{|(Q0)(z)|}{\ell(z)} \leq \frac{|c(z)|}{\ell(z)} + \beta \int \frac{|r(z')|}{\ell(z)} P(z, dz') \leq (1 + \beta)/m' < \infty$$

for all $z \in \mathsf{Z}$, so $\|Q0\|_\ell < \infty$. The measurability of $Q\psi$ follows immediately from our primitive assumptions. Hence, condition (b) holds. By the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3), we have

$$\int \mathbb{E}_{z'} |r(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |r(Z_{t+1})| \;\text{ and }\; \int \mathbb{E}_{z'} |c(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |c(Z_{t+1})|.$$

Let $h(z) := \sum_{t=1}^{n-1} \mathbb{E}_z|r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z|c(Z_t)|$, then we have

$$\int h(z')P(z,\mathrm{d}z') = \sum_{t=2}^{n} \mathbb{E}_z|r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z|c(Z_t)|. \tag{24}$$

By the construction of $m'$ and $d'$, we have $m + d_1m' > 1$ and $(d_2m' + d + d')/(m + d_1m') \leq d'$. Assumption 3.1 and (24) then imply that

$$\begin{aligned}\int \kappa(z')P(z,dz') &= m' \sum_{t=1}^{n} \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + \int g(z')P(z,dz') + d' \\ &\leq m' \sum_{t=1}^{n-1} \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + (m + d_1m')g(z) + d_2m' + d + d' \\ &\leq (m + d_1m') \left( \frac{m'}{m + d_1m'}\, h(z) + g(z) + d' \right) \leq (m + d_1m')\ell(z). \end{aligned} \tag{25}$$

Since $\ell \leq \kappa$, this implies that for all $z \in \mathsf{Z}$, we have

$$\int \kappa(z')P(z,\mathrm{d}z') \leq (m + d_1m')\kappa(z) \quad \text{and} \quad \int \ell(z')P(z,\mathrm{d}z') \leq (m + d_1m')\ell(z). \tag{26}$$

Hence, for all $\psi \in b_\ell\mathsf{Z}$, $a \in \mathbb{R}_+$ and $z \in \mathsf{Z}$, we have

$$\begin{aligned} Q(\psi + a\ell)(z) &= c(z) + \beta \int \max\left\{r(z'), \psi(z') + a\ell(z')\right\} P(z,dz') \\ &\leq c(z) + \beta \int \max\left\{r(z'), \psi(z')\right\} P(z,dz') + a\beta \int \ell(z')P(z,dz') \\ &\leq Q\psi(z) + a\beta(m + d_1m')\ell(z) = Q\psi(z) + a\rho\ell(z). \end{aligned}$$

So condition (c) holds, and $Q : b_\ell\mathsf{Z} \to b_\ell\mathsf{Z}$ is a contraction mapping of modulus $\rho$.

Moreover, lemma 5.5 and the analysis related to (5) imply that $\psi^*$ is indeed a fixed point of $Q$ under assumption 3.1. Lemma 5.4 implies that $\psi^* \in b_\ell\mathsf{Z}$. Hence, $\psi^*$ must coincide with the unique fixed point of $Q$ under $b_\ell\mathsf{Z}$, and claim (1) holds.

The proof of claim (2) is similar. In particular, using (26) one can show that $T : b_\kappa\mathsf{Z} \to b_\kappa\mathsf{Z}$ is a contraction mapping with the same modulus. We omit the details. □

*Proof of proposition 3.4.* Let $b_\ell c\mathsf{Z}$ be the set of continuous functions in $b_\ell\mathsf{Z}$. Since $\ell$ is continuous by assumption 3.2, $b_\ell c\mathsf{Z}$ is a closed subset of $b_\ell\mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of $\psi^*$, it suffices to verify that $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For fixed $\psi \in b_\ell c\mathsf{Z}$, let $h(z) := \max\{r(z), \psi(z)\}$, then there exists $G \in \mathbb{R}_+$ such that $|h(z)| \leq |r(z)| + G\ell(z) =: \tilde{h}(z)$. By assumption 3.2,

$z \mapsto \tilde{h}(z) \pm h(z)$ are nonnegative and continuous. For all $z \in \mathsf{Z}$ and $\{z_m\} \subset \mathsf{Z}$ with $z_m \to z$, the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1) implies that

$$\int \left(\tilde{h}(z') \pm h(z')\right) P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \int \left(\tilde{h}(z') \pm h(z')\right) P(z_m, \mathrm{d}z').$$

Since $\lim_{m\to\infty} \int \tilde{h}(z')P(z_m, \mathrm{d}z') = \int \tilde{h}(z')P(z, \mathrm{d}z')$ by assumption 3.2, we have

$$\pm \int h(z')P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \left( \pm \int h(z')P(z_m, \mathrm{d}z') \right),$$

where we have used the fact that for all sequences $\{a_m\}$ and $\{b_m\}$ in $\mathbb{R}$ with $\lim\limits_{m\to\infty} a_m$ exists, we have: $\liminf\limits_{m\to\infty}(a_m + b_m) = \lim\limits_{m\to\infty} a_m + \liminf\limits_{m\to\infty} b_m$. Hence,

$$\limsup_{m\to\infty} \int h(z')P(z_m, \mathrm{d}z') \leq \int h(z')P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int h(z')P(z_m, \mathrm{d}z'), \tag{27}$$

i.e., $z \mapsto \int h(z')P(z, \mathrm{d}z')$ is continuous. Since $c$ is continuous by assumption, $Q\psi \in b_\ell c\mathsf{Z}$. Hence, $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ and $\psi^*$ is continuous, as was to be shown. The continuity of $v^*$ follows from the continuity of $\psi^*$ and $r$ and the fact that $v^* = r \vee \psi^*$. $\square$

*Proof of proposition 3.3.* Let $\mathscr{C} := b_\ell \mathsf{Z}$ and $\mathcal{V} := b_\kappa \mathsf{Z}$. Let $\mathscr{V} := R\mathscr{C}$ and $\mathcal{C} := L\mathcal{V}$ as defined respectively in (16) and (8). Our first goal is to show that $\mathscr{V} \subset b_\kappa \mathsf{Z}$ and $\mathcal{C} \subset b_\ell \mathsf{Z}$.

For all $v \in \mathscr{V}$, by definition, there exists a $\psi \in \mathscr{C}$ such that $v = R\psi = r \vee \psi$. Since $\mathscr{C} = b_\ell \mathsf{Z}$, we have $|\psi| \leq M\ell$ for some constant $M < \infty$. Without loss of generality, we can let $M > 1/m'$, where $m'$ is defined as in theorem 3.1 (see (23) in the proof of theorem 3.1). Hence, $|v| \leq |r| + |\psi| \leq M(m'|r| + \ell) = M\kappa$, i.e., $\|v\|_\kappa < \infty$. Moreover, $v$ is measurable since both $r$ and $\psi$ are. Hence, $v \in b_\kappa \mathsf{Z}$. Since $v$ is arbitrary, we have $\mathscr{V} \subset b_\kappa \mathsf{Z}$.

For all $\psi \in \mathcal{C}$, by definition, there exists $v \in \mathcal{V}$ such that $\psi = Lv = c + \beta P v$. Since $\mathcal{V} = b_\kappa \mathsf{Z}$, we have $|v| \leq M\kappa$ for some constant $M < \infty$. By (25) in the proof of theorem 3.1, $|\psi| \leq |c| + \|v\|_\kappa \ell \leq (1/m' + \|v\|_\kappa)\ell$, i.e., $\|\psi\|_\ell < \infty$. Moreover, $\psi$ is measurable by our primitive assumptions. Hence, $\psi \in b_\ell \mathsf{Z}$. Since $\psi$ is arbitrary, we have $\mathcal{C} \subset b_\ell \mathsf{Z}$.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 5.3 and theorem 3.1, we have

$$\left| Q^{t+1}\psi(z) - \psi^*(z) \right| = \left| LT^t R\psi(z) - Lv^*(z) \right| = \beta \left| P(T^t R\psi)(z) - Pv^*(z) \right|.$$

Since we have shown in the proof of theorem 3.1 that $\int \kappa(z')P(z, \mathrm{d}z') \leq (m + m'd_1)\ell(z)$ for all $z \in \mathsf{Z}$ (see equation (25)), by the definition of operator $P$, for all $z \in \mathsf{Z}$, we have

$$\begin{aligned}
\left| P(T^t R\psi)(z) - Pv^*(z) \right| &\leq \int \left| (T^t R\psi)(z') - v^*(z') \right| P(z, \mathrm{d}z') \\
&\leq \|T^t R\psi - v^*\|_\kappa \int \kappa(z')P(z, \mathrm{d}z') \leq (m + m'd_1)\|T^t R\psi - v^*\|_\kappa \, \ell(z).
\end{aligned}$$

Recall $\rho := \beta(m + m'd_1) < 1$ defined in (23). The above results imply that

$$\|Q^{t+1}\psi - \psi^*\|_\ell \leq \beta(m + m'd_1) \left\|T^t R\psi - v^*\right\|_\kappa = \rho \left\|T^t R\psi - v^*\right\|_\kappa$$

for all $\psi \in \mathscr{C}$. Hence, claim (1) is verified.

Regarding claim (2), for all $v \in \mathcal{V}$, propositions 3.1–3.2 and theorem 3.1 imply that

$$\left|T^{t+1}v(z) - v^*(z)\right| = \left|(RQ^t L)v(z) - R\psi^*(z)\right| \leq \left|Q^t Lv(z) - \psi^*(z)\right| \leq \|Q^t Lv - \psi^*\|_\ell \, \ell(z)$$

for all $z \in \mathsf{Z}$, where the first inequality is due to the elementary fact that $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$ for all $a, b, c, d \in \mathbb{R}$. Since $\ell \leq \kappa$ by construction, we have

$$\frac{\left|T^{t+1}v(z) - v^*(z)\right|}{\kappa(z)} \leq \frac{\left|T^{t+1}v(z) - v^*(z)\right|}{\ell(z)} \leq \left\|Q^t Lv - \psi^*\right\|_\ell$$

for all $z \in \mathsf{Z}$. Hence, $\left\|T^{t+1}v - v^*\right\|_\kappa \leq \left\|Q^t Lv - \psi^*\right\|_\ell$ and claim (2) holds. □

*Proof of theorem 3.2.* Since $r, c \in L_q(\pi)$, by the monotonicity of the $L_p$-norm, we have $r, c \in L_p(\pi)$ for all $1 \leq p \leq q$. Our first goal is to prove claim (1).

**Step 1.** We show that $Q\psi \in L_p(\pi)$ for all $\psi \in L_p(\pi)$. Notice that for all $z \in \mathsf{Z}$,

$$\begin{aligned}
|Q\psi(z)|^p &\leq 2^p |c(z)|^p + (2\beta)^p \left[\int \left|r(z')\right| \vee \left|\psi(z')\right| P(z, \mathrm{d}z')\right]^p \\
&\leq 2^p |c(z)|^p + (2\beta)^p \int \left[\left|r(z')\right| \vee \left|\psi(z')\right|\right]^p P(z, \mathrm{d}z') \\
&\leq 2^p |c(z)|^p + (2\beta)^p \left(\int \left|r(z')\right|^p P(z, \mathrm{d}z') + \int \left|\psi(z')\right|^p P(z, \mathrm{d}z')\right),
\end{aligned}$$

where for the first and the third inequality we have used the elementary fact that $(a+b)^p \leq 2^p (a \vee b)^p \leq 2^p (a^p + b^p)$ for all $a, b, p \geq 0$, and the second inequality is based on Jensen's inequality. Then we have $\|Q\psi\|_p < \infty$, since the above result implies that

$$\begin{aligned}
\int |Q\psi(z)|^p \, \pi(\mathrm{d}z) &\leq 2^p \int |c(z)|^p \pi(\mathrm{d}z) + (2\beta)^p \int \int \left|r(z')\right|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&\quad + (2\beta)^p \int \int \left|\psi(z')\right|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&= 2^p \int |c(z)|^p \pi(\mathrm{d}z) + (2\beta)^p \int \left|r(z')\right|^p \pi(\mathrm{d}z') + (2\beta)^p \int \left|\psi(z')\right|^p \pi(\mathrm{d}z') \\
&= 2^p \|c\|_p^p + (2\beta)^p \|r\|_p^p + (2\beta)^p \|\psi\|_p^p < \infty,
\end{aligned}$$

where the first equality follows from the Fubini theorem and the fact that $\pi$ is a stationary distribution. We have thus verified that $Q\psi \in L_p(\pi)$.

**Step 2.** We show that $Q$ is a contraction mapping on $\left(L_p(\pi), \|\cdot\|_p\right)$ of modulus $\beta$. For all $\psi, \phi \in L_p(\pi)$, we have

$$\begin{aligned}
|Q\psi(z) - Q\phi(z)|^p &= \beta^p \left| \int \left[ r(z') \vee \psi(z') - r(z') \vee \phi(z') \right] P(z, \mathrm{d}z') \right|^p \\
&\leq \beta^p \int \left| r(z') \vee \psi(z') - r(z') \vee \phi(z') \right|^p P(z, \mathrm{d}z') \\
&\leq \beta^p \int \left| \psi(z') - \phi(z') \right|^p P(z, \mathrm{d}z'),
\end{aligned}$$

where the first inequality holds by Jensen's inequality, and the second follows from the elementary fact that $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$ for all $a, b, c, d \in \mathbb{R}$. Hence,

$$\begin{aligned}
\int |Q\psi(z) - Q\phi(z)|^p \, \pi(\mathrm{d}z) &\leq \beta^p \int \int \left| \psi(z') - \phi(z') \right|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&= \beta^p \int \left| \psi(z') - \phi(z') \right|^p \pi(\mathrm{d}z'),
\end{aligned}$$

and we have $\|Q\psi - Q\phi\|_p \leq \beta \|\psi - \phi\|_p$. Thus, $Q$ is a contraction on $L_p(\pi)$ of modulus $\beta$.

Since $\left(L_p(\pi), \|\cdot\|_p\right)$ is a Banach space, based on the contraction mapping theorem, $Q$ admits a unique fixed point in $L_p(\pi)$. In order to prove claim (1), it remains to show that $\psi^* \in L_p(\pi)$ and that $\psi^*$ is a fixed point of $Q$.

**Step 3.** We show that $\psi^*, v^* \in L_p(\pi)$. Since $|\psi^*(z)| \vee |v^*(z)| \leq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]$, we have

$$\left[ \int |\psi^*(z)|^p \pi(\mathrm{d}z) \right] \vee \left[ \int |v^*(z)|^p \pi(\mathrm{d}z) \right] \leq \int \left( \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right)^p \pi(\mathrm{d}z). \quad (28)$$

Since $\pi$ is stationary, the Fubini theorem implies that

$$\int \mathbb{E}_z |r(Z_t)|^p \pi(\mathrm{d}z) = \int \int |r(z')|^p P^t(z, \mathrm{d}z') \pi(\mathrm{d}z) = \int |r(z')|^p \pi(\mathrm{d}z') = \|r\|_p^p.$$

Similarly, we have $\int \mathbb{E}_z |c(Z_t)|^p \pi(\mathrm{d}z) = \|c\|_p^p$. The Minkowski and Jensen inequalities then imply that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p &\leq \sum_{t=0}^{n} \beta^t \left\| \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p \\
&\leq \sum_{t=0}^{n} \beta^t \left[ \int \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]^p \pi(\mathrm{d}z) \right]^{1/p} \\
&\leq \sum_{t=0}^{n} \beta^t \left[ \int \left( \mathbb{E}_z |r(Z_t)|^p + \mathbb{E}_z |c(Z_t)|^p \right) \pi(\mathrm{d}z) \right]^{1/p} \\
&= \sum_{t=0}^{n} \beta^t \left( \|r\|_p^p + \|c\|_p^p \right)^{1/p} \leq \frac{\left( \|r\|_p^p + \|c\|_p^p \right)^{1/p}}{1-\beta} < \infty, \qquad (29)
\end{aligned}$$

Moreover, by the monotone convergence theorem, we have

$$\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p \to \left\| \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p . \qquad (30)$$

Together, (29)–(30) imply that $\left\| \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|] \right\|_p < \infty$. By (28), we have $\|\psi^*\|_p \vee \|v^*\|_p < \infty$ and thus $\psi^*, v^* \in L_p(\pi)$.

**Step 4.** We show that $v^*$ is a fixed point of $T$ and $\psi^*$ is a fixed point of $Q$, i.e., $\|Tv^* - v^*\|_p = 0$ and $\|Q\psi^* - \psi^*\|_p = 0$. It suffices to show that $Tv^* = v^*$ and $Q\psi^* = \psi^*$ with probability one (measured by $\pi$, same below). By theorem 1.11 of Peskir and Shiryaev (2006) and the analysis related to (5), it suffices to show that $\mathbb{E}_z \left( \sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ with probability one. This obviously holds since the left-hand-side term is dominated by $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]$, which is finite with probability one since in step 3 we have shown that it is an object of $L_p(\pi)$.

Steps 1–4 imply that claim (1) holds. The proof of claim (2) is similar and thus omitted. $\square$

*Proof of proposition 3.5.* Let $\mathscr{C} = \mathcal{V} := L_p(\pi)$, and let $\mathscr{V} := R\mathscr{C}$ and $\mathcal{C} := L\mathcal{V}$ as defined respectively in (16) and (8). Our first goal is to show that $\mathscr{V}, \mathcal{C} \subset L_p(\pi)$.

For all $v \in \mathscr{V}$, there exists a $\psi \in \mathscr{C}$ such that $v = R\psi = r \vee \psi$. Since $\mathscr{C} = L_p(\pi)$ and $r \in L_p(\pi)$ by assumption 3.3, we have $v \in L_p(\pi)$. Hence, $\mathscr{V} \subset L_p(\pi)$. For all $\psi \in \mathcal{C}$, there exists $v \in \mathcal{V}$ such that $\psi = Lv = c + \beta P v$. Since $\mathcal{V} = L_p(\pi)$ and $\pi$ is stationary, Jensen's inequality implies that $Pv \in L_p(\pi)$. Since $c \in L_p(\pi)$, Minkowski's inequality then implies that $\psi \in L_p(\pi)$. Hence, $\mathcal{C} \subset L_p(\pi)$, as claimed.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 5.3, theorem 3.2, Jensen's inequality and Fubini's theorem, we have

$$\begin{aligned}
\left\| Q^{t+1}\psi - \psi^* \right\|_p &= \left[ \int \left| Q^{t+1}\psi(z) - \psi^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&= \left[ \int \left| LT^t R\psi(z) - Lv^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&= \beta \left[ \int \left| PT^t R\psi(z) - Pv^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&\leq \beta \left[ \int \int \left| T^t R\psi(z') - v^*(z') \right|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \right]^{1/p} \\
&= \beta \left[ \int \left| T^t R\psi(z') - v^*(z') \right|^p \pi(\mathrm{d}z') \right]^{1/p} = \beta \left\| T^t R\psi - v^* \right\|_p.
\end{aligned}$$

Regarding claim (2), for all $v \in \mathcal{V}$, based on propositions 3.1–3.2 and theorem 3.2, we have

$$\begin{aligned}
\left\| T^{t+1}v - v^* \right\|_p &= \left[ \int \left| T^{t+1}v(z) - v^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&= \left[ \int \left| RQ^t Lv(z) - R\psi^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&\leq \left[ \int \left| Q^t Lv(z) - \psi^*(z) \right|^p \pi(\mathrm{d}z) \right]^{1/p} = \left\| Q^t Lv - \psi^* \right\|_p.
\end{aligned}$$

Hence, the second claim holds. This concludes the proof. □

*Proof of lemma 5.1.* Recall that if $X \sim LN(\mu, \sigma^2)$, then $\mathbb{E}\, X^s = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ for all $s \in \mathbb{R}$. By (14), the distribution of $\theta_n$ given $\theta_0 = \theta$ follows $LN\left(\rho^n \ln\theta,\, \sigma^2 \sum_{i=0}^{n-1} \rho^{2i}\right)$. Hence, $\mathbb{E}_{\,\theta}\, \theta_n = \theta^{\rho^n} \exp\left[\frac{\sigma^2(1-\rho^{2n})}{2(1-\rho^2)}\right]$.

Since $w = \eta + \theta\xi$ and $|\ln w| \leq 1/w + w$, we have $|\ln w_n| \leq \eta_n^{-1} + \eta_n + \theta_n\xi_n$. Hence,

$$\mathbb{E}_{\,\theta} |\ln w_n| \leq \mathbb{E}_{\,\theta} \left( \eta_n^{-1} + \eta_n + \theta_n\xi_n \right) = \mu_1 + \mu_2 + \mu_3\, \mathbb{E}_{\,\theta}\theta_n = A_n\theta^{\rho^n} + B, \tag{31}$$

where $\mu_1$, $\mu_2$ and $\mu_3$ are respectively the mean of $\eta_n^{-1}$, $\eta_n$ and $\xi_n$, $B := \mu_1 + \mu_2$ and $A_n := \mu_3 \exp\left[\frac{\sigma^2(1-\rho^{2n})}{2(1-\rho^2)}\right]$. Claim (a) is verified.

To verify claim (b), consider a sequence $\theta^{(m)} \to \theta$. By the Fatou's lemma,

$$\mathbb{E}_{\,\theta} \left( \eta_n^{-1} + \eta_n + \theta_n\xi_n \pm |\ln w_n| \right) \leq \liminf_m \mathbb{E}_{\,\theta^{(m)}} \left( \eta_n^{-1} + \eta_n + \theta_n\xi_n \pm |\ln w_n| \right).$$

Since $\theta \mapsto \mathbb{E}_{\,\theta} \left( \eta_n^{-1} + \eta_n + \theta_n\xi_n \right)$ is continuous by (31), the above inequality yields

$$\pm\mathbb{E}_{\,\theta} |\ln w_n| \leq \liminf_m \left( \pm\mathbb{E}_{\,\theta^{(m)}} |\ln w_n| \right),$$

i.e., $\lim_m \mathbb{E}_{\theta^{(m)}} |\ln w_n| = \mathbb{E}_\theta |\ln w_n|$. Hence, $\theta \mapsto \mathbb{E}_\theta |\ln w_n|$ is continuous, as claimed. □

*Proof of lemma 5.2.* Since $w = \eta + \theta\xi$ and $|\ln w| \leq w + 1/w$, we have $|\ln w|^q \leq 3^q(\eta^{-q} + \eta^q + \theta^q \xi^q)$. By the assumption on $\{\eta_t\}$ and $\{\xi_t\}$, taking expectation (w.r.t $\pi$) yields

$$\mathbb{E} |\ln w|^q \leq 3^q (\mathbb{E}\, \eta^{-q} + \mathbb{E}\, \eta^q + \mathbb{E}\, \xi^q \, \mathbb{E}\, \theta^q) < \infty.$$

Since $r(w) = \ln w/(1 - \beta)$, this inequality implies $r \in L_q(\pi)$. Moreover, $c \in L_q(\pi)$ is trivial since $c$ is constant. □

## Appendix C: Proof of Table 1 Results

To prove the results of table 1, we introduce some elementary facts on time complexity:

(a) The multiplication of an $m \times n$ matrix and an $n \times p$ matrix has complexity $\mathcal{O}(mnp)$. See, for example, section 2.5.4 of Skiena (2008).
(b) The binary search algorithm finds the index of an element in a given sorted array of length $n$ in $\mathcal{O}(\log(n))$ time. See, for example, section 4.9 of Skiena (2008).

For finite space (FS) approximation, time complexity of VFI (1-loop) reduces to the complexity of matrix multiplication $\hat{P}\vec{v}$, which is of order $\mathcal{O}(K^2M^2)$ based on the shape of $\hat{P}$ and $\vec{v}$ and fact (a) above. Similarly, time complexity of CVI (1-loop) is determined by $\hat{F}(\vec{r} \vee \vec{\psi})$, which has complexity $\mathcal{O}(KM^2)$. The $n$-loop complexity is scaled up by $\mathcal{O}(n)$.

For the infinite space (IS) case, let $\mathcal{O}(g)$ and $\mathcal{O}(h)$ denote respectively the complexity (of single point evaluation) of the interpolating functions $g$ and $h$. Counting the floating point operations associated with all grid points inside the inner loops shows that the one step complexities of VFI and CVI are $\mathcal{O}(NKM)\mathcal{O}(g)$ and $\mathcal{O}(NM)\mathcal{O}(h)$, respectively. Since binary search function evaluation is adopted for the indicated function interpolation mechanisms (see table 1 note), and in particular, since evaluating $g$ at a given point uses binary search $\ell + n$ times, based on fact (b) above, we have

$$\mathcal{O}(g) = \mathcal{O}\left(\sum_{i=1}^{\ell} \log(K_i) + \sum_{j=1}^{n} \log(M_j)\right) = \mathcal{O}(\log(KM)).$$

Similarly, we can show that $\mathcal{O}(h) = \mathcal{O}(\log(M))$. Combining these results, we see that the claims of the IS case hold, concluding our proof of table 1 results.

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