

# ECON-GA 1025 Macroeconomic Theory I

## Lecture 13

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# Today's Lecture

- Lots of new theory
- Just joking
- Euler equation + revision
- See “Lecture 14” for general DP results (not examinable)

# Optimal Savings: The Envelope Condition

Recall the IID model with Bellman equation

$$v(y) = \max_{0 \leq c \leq y} \left\{ u(c) + \beta \int v(f(y-c)z) \varphi(dz) \right\} \quad (y \in \mathbb{R}_+)$$

We know that

$$\sigma \text{ is optimal} \iff \sigma \text{ is } v^*\text{-greedy}$$

We can get additional characterizations of optimality if we impose more conditions

**Assumption.** (INA) Both  $f$  and  $u$  are strictly increasing, continuously differentiable and strictly concave

In addition,

$$f(0) = 0, \quad \lim_{k \rightarrow 0} f'(k) > 0$$

and

$$u(0) = 0, \quad \lim_{c \rightarrow 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \rightarrow \infty} u'(c) = 0$$

**Remark.** We ignore the restriction  $u(0) = 0$  in some applications below — I'm aiming to remove it

**Proposition.** Let

- $v$  be an increasing concave function in  $bc\mathbb{R}_+$
- $\sigma$  be the unique  $v$ -greedy policy in  $\Sigma$

If assumption (INA) holds, then

1.  $\sigma$  is interior, while
2.  $Tv$  is continuously differentiable and satisfies

$$(Tv)' = u' \circ \sigma$$

**Corollary** If  $\sigma^*$  is the optimal consumption policy, then

$$(v^*)' = u' \circ \sigma^*$$

Proof that  $(Tv)' = u' \circ \sigma$  when  $\sigma$  is  $v$ -greedy:

Since  $\sigma$  is  $v$ -greedy,

$$Tv(y) = u(\sigma(y)) + \beta \int v(f(y - \sigma(y))z) \varphi(dz)$$

By the **envelope theorem**,

$$(Tv)'(y) = \beta \int (v)'(f(y - \sigma(y))z) f'(y - \sigma(y))z \varphi(dz)$$

The FOC from the Bellman equation yields

$$u'(\sigma(y)) = \beta \int (v)'(f(y - \sigma(y))z) f'(y - \sigma(y))z \varphi(dz)$$

Combining the last two equations gives  $(Tv)' = u' \circ \sigma$

Let  $\mathcal{C} :=$  all continuous strictly increasing  $\sigma \in \Sigma$  satisfying

$$0 < \sigma(y) < y \text{ for all } y > 0$$

We say that  $\sigma \in \mathcal{C}$  **satisfies the Euler equation** if

$$(u' \circ \sigma)(y) = \beta \int (u' \circ \sigma)(f(y - \sigma(y))z) f'(y - \sigma(y))z \varphi(dz)$$

for all  $y > 0$

- A functional equation in **policies**

In sequence notation,  $u'(c_t) = \beta \mathbb{E} u'(c_{t+1}) f'(k_t) z_{t+1}$

Let's introduce an operator  $K$  corresponding to the Euler equation

Fix  $\sigma \in \mathcal{C}$  and  $y > 0$

The value  $K\sigma(y)$  is the  $c$  in  $(0, y)$  that solves

$$u'(c) = \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c)z \varphi(dz)$$

We call  $K$  the **Coleman–Reffett** operator

**Ex.** Show  $\sigma$  in  $\mathcal{C}$  is a fixed point of  $K$  if and only if it satisfies the Euler equation



Proof that  $K$  is well defined:

For any  $\sigma \in \mathcal{C}$ , the RHS of

$$u'(c) = \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c)z \varphi(dz)$$

is continuous, strictly increasing in  $c$ , diverges to  $+\infty$  as  $c \uparrow y$

The LHS is continuous, strictly decreasing in  $c$ , diverges to  $+\infty$  as  $c \downarrow 0$

Hence

$$H(y, c) := u'(c) - \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c)z \varphi(dz)$$

when regarded as a function of  $c$ , has exactly one zero

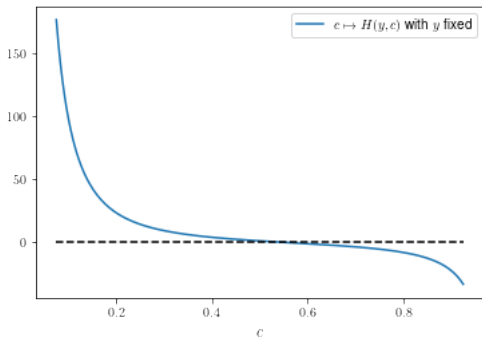


Figure: Solving for the  $c$  that satisfies  $H(y, c) = 0$ .

# The Euler Equation and Optimality

**Proposition.** If assumption (INA) holds and  $\sigma^*$  is the unique optimal policy, then

1.  $(\mathcal{C}, K)$  is globally stable and
2. the unique fixed point of  $K$  in  $\mathcal{C}$  is  $\sigma^*$

In particular,  $\sigma \in \mathcal{C}$  is optimal if and only if it satisfies the Euler equation

Sketch of proof:

Let  $\mathcal{V}$  be all strictly concave, continuously differentiable  $v$  mapping  $\mathbb{R}_+$  to itself and satisfying  $v(0) = 0$  and  $v'(y) > u'(y)$  whenever  $y > 0$

As before, let  $\mathcal{C}$  be all a continuous, strictly increasing functions on  $\mathbb{R}_+$  satisfying  $0 < \sigma(y) < y$

For  $v \in \mathcal{V}$  let  $Mv$  be defined by

$$(Mv)(y) = \begin{cases} m(v'(y)) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases} \quad (1)$$

where  $m(y) := (u')^{-1}(y)$

The course notes show that

1.  $M$  is a homeomorphism from  $\mathcal{V}$  to  $\mathcal{C}$
2. for every increasing concave function in  $bc\mathbb{R}_+$ ,

$$\sigma := MTv$$

is the unique  $v$ -greedy policy

3. The Bellman operator and Coleman–Reffett operator are related by

$$T = M^{-1} \circ K \circ M \text{ on } \mathcal{V}$$

**Ex.** Use 1–3 above to show that  $(\mathcal{C}, K)$  is globally stable with unique fixed point  $\sigma^*$

**Remark.** The Euler equation is often paired with the **transversality condition**

$$\lim_{t \rightarrow \infty} \beta^t \mathbb{E} u'(c_t) k_t = 0$$

Standard results (see, e.g., Stokey and Lucas) tell us that

Euler + transversality condition  $\implies$  optimality

Our last result shows transversality is not needed under our assumptions

**Ex.** Following the basic CRRA cake eating model, set

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad f(k)z = Rk$$

Insert the conjecture  $\sigma^*(y) = \theta y$  into the Euler equation

Recover our earlier result that this policy is optimal when

$$\theta = 1 - \left(\beta R^{1-\gamma}\right)^{1/\gamma}$$

**Ex.** Repeat for the log / CD model, where  $u(c) = \ln c$  and

$$f(k)z = Ak^\alpha z, \quad 0 < A, \quad 0 < \alpha < 1$$

Insert the conjecture  $\sigma^*(y) = \theta y$  into the Euler equation and recover your earlier result for the optimal policy

## Exercise: Predicting Quadratics

Let

- $\{\mathcal{G}_t\}_{t \geq 1}$  be a filtration
- $\{\zeta_t\}_{t \geq 1}$  be a stochastic process in  $\mathbb{R}^J$

Recall:  $\{\zeta_t\}_{t \geq 1}$  is called a **martingale difference sequence** (MDS) with respect to  $\{\mathcal{G}_t\}$  if

- $\mathbb{E}\|\zeta_t\|_1 < \infty$  and
- $\{\zeta_t\}_{t \geq 1}$  is adapted to  $\{\mathcal{G}_t\}$
- and

$$\mathbb{E}[\zeta_{t+1} | \mathcal{G}_t] = 0, \quad \forall t \geq 1$$



Suppose that

- $x_{t+1} = Ax_t + C\tilde{\zeta}_{t+1}$  in  $\mathbb{R}^n$  with  $x_0$  given
- $\mathcal{G}_t = \{x_0, \zeta_0, \zeta_1, \dots, \zeta_t\}$
- $\{\tilde{\zeta}_t\}_{t \geq 1}$  is an  $\mathbb{R}^j$ -valued MDS with respect to  $\mathcal{G}_t$  satisfying

$$\mathbb{E}[\tilde{\zeta}_t \tilde{\zeta}_t'] = I$$

Question: Is  $\{x_t\}$  adapted to  $\mathcal{G}_t$ ?

**Ex.** Let  $\mathbb{E}_t := \mathbb{E}[\cdot | \mathcal{G}_t]$

Show that if  $H \in \mathcal{M}(n \times n)$ , then

$$\mathbb{E}_t[x'_{t+1} H x_{t+1}] = x'_t A' H A x_t + \text{trace}(C' H C)$$

Solution: We have

$$\mathbb{E}_t[x'_{t+1}Hx_{t+1}] = \mathbb{E}_t[(Ax_t + C\tilde{\zeta}_{t+1})'H(Ax_t + C\tilde{\zeta}_{t+1})]$$

The RHS expands to

$$\begin{aligned} \mathbb{E}_t[x'_t A' H A x_t] + 2\mathbb{E}_t[x'_t A' H C \tilde{\zeta}_{t+1}] + \mathbb{E}_t[\tilde{\zeta}'_{t+1} C' H C \tilde{\zeta}_{t+1}] \\ = I + II + III \end{aligned}$$

Since  $x_t$  is known at  $t$  we have

$$I = \mathbb{E}_t[x_t' A' H A x_t] = x_t' A' H A x_t$$

Since  $\{\zeta_t\}$  is an MDS,

$$II = 2\mathbb{E}_t[x_t' A' H C \zeta_{t+1}] = 2x_t' A' H C \mathbb{E}_t[\zeta_{t+1}] = 0$$

Finally,

$$III = \mathbb{E}_t[\zeta_{t+1}' C' H C \zeta_{t+1}] = \text{trace}(C' H C)$$

Hence

$$\mathbb{E}_t[x_{t+1}' H x_{t+1}] = x_t' A' H A x_t + \text{trace}(C' H C)$$

# Application: LQ Risk Neutral Asset Pricing

Recall the risk neutral asset pricing formula

$$p_t = \beta \mathbb{E}_t[d_{t+1} + p_{t+1}]$$

Here

- $\{d_t\}$  is a cash flow
- $p_k$  is asset price at time  $k$
- $\beta \in (0, 1)$  discounts values
- $\mathbb{E}_t$  is time  $t$  conditional expectation

Aim: solve for  $\{p_t\}$

Assume that

$$d_t = x_t' D x_t \text{ for some positive definite } D$$

Here

- $x_{t+1} = Ax_t + C\tilde{\zeta}_{t+1}$  in  $\mathbb{R}^n$  with  $x_0$  given
- $\mathcal{G}_t = \{x_0, \zeta_0, \zeta_1, \dots, \zeta_t\}$
- $\{\tilde{\zeta}_t\}_{t \geq 1}$  is an  $\mathbb{R}^j$ -valued MDS with respect to  $\mathcal{G}_t$  satisfying

$$\mathbb{E}[\tilde{\zeta}_t \tilde{\zeta}_t'] = I$$

# Prices as Functions of the State

We conjecture that

$$p_t = p(x_t) \quad \text{for some function } p$$

Another leap: guess that prices are a **quadratic** in  $x_t$

In particular, we guess that

$$p(x) = x'Px + \delta$$

for some positive definite  $P$  and scalar  $\delta$

Substituting

$$p_t = x_t' P x_t + \delta \quad \text{and} \quad d_t = x_t' D x_t$$

into

$$p_t = \beta \mathbb{E}_t [d_{t+1} + p_{t+1}]$$

gives

$$\begin{aligned} x_t' P x_t + \delta &= \beta \mathbb{E}_t [x_{t+1}' D x_{t+1} + x_{t+1}' P x_{t+1} + \delta] \\ &= \beta \mathbb{E}_t [x_{t+1}' (D + P) x_{t+1}] + \beta \delta \\ &= \beta x_t' A' (D + P) A x_t + \beta \text{trace}(C' (D + P) C) + \beta \delta \end{aligned}$$

So, we seek a pair  $P \in \mathcal{M}(n \times n)$ ,  $\delta \in \mathbb{R}$  such that

$$x'Px + \delta = \beta x' A'(D + P)Ax + \beta \text{trace}(C'(D + P)C) + \beta\delta$$

for any  $x \in \mathbb{R}^n$

Claim: If  $P^*$  satisfies

$$P^* = \beta A'(D + P^*)A$$

and

$$\delta^* := \frac{\beta}{1 - \beta} \text{trace}(C'(D + P^*)C)$$

then  $P^*, \delta^*$  solves the above equation for any  $x$



Proof: By hypothesis,  $P^* = \beta A'(D + P^*)A$

$$\therefore x'P^*x = \beta x'A'(D + P^*)Ax$$

$$\therefore x'P^*x + \delta^* = \beta x'A'(D + P^*)Ax + \delta^*$$

To complete the proof, suffices to show that

$$\delta^* = \beta \text{trace}(C'(D + P^*)C) + \beta \delta^*$$

True by definition of  $\delta^*$

Summary: With such a  $P^*$ ,

$$p_t := x_t' P^* x_t + \delta^*$$

is an equilibrium price sequence

But does there exist a  $P \in \mathcal{M}(n \times n)$  that solves

$$P = \beta A'(D + P)A$$

**Ex.** Under what condition does a unique solution exist?

Solution: Write  $P = \beta A'(D + P)A$  as the discrete Lyapunov equation

$$P = \Lambda' P \Lambda + M$$

where

- $\Lambda := \sqrt{\beta}A$
- $M := \Lambda'D\Lambda$

The solution  $P^*$  is the fixed point of operator  $\ell$  defined by

$$\ell P = \Lambda' P \Lambda + M$$

As shown previously,  $\ell$  is globally stable when  $r(\Lambda) < 1$

Equivalent:  $r(A) \leq 1/\sqrt{\beta}$

Recall that  $P^*$  is the solution to

$$P = \beta A'(D + P)A$$

**Ex.** Show that  $P$  is positive semidefinite

We need to show that  $x'P^*x \geq 0$  for all  $x \in \mathbb{R}^n$

Let  $\mathcal{M}_P$  be the set of positive semidefinite matrices in  $\mathcal{M}(n \times n)$

This set is closed in  $\mathcal{M}(n \times n)$  under the matrix norm

To see this, pick

- any  $\{E_n\} \subset \mathcal{M}_P$  and  $E \in \mathcal{M}(n \times n)$  with  $E_n \rightarrow E$
- any  $x \in \mathbb{R}^n$

We showed in another context that  $x'E_nx \rightarrow x'Ex$  in  $\mathbb{R}$

Since  $x'E_nx \geq 0$  for all  $n$ , we have  $x'Ex \geq 0$

Since  $x$  was arbitrary we have  $E \in \mathcal{M}_P$

Hence  $\mathcal{M}_P$  is closed

Since

1.  $\mathcal{M}_P$  is closed in  $\mathcal{M}(n \times n)$
2.  $P^*$  is the fixed point of  $\ell P = \Lambda' P \Lambda + M$

it suffices to show that  $\ell$  maps  $\mathcal{M}_P$  to itself

So pick any  $P \in \mathcal{M}_P$  and any  $x \in \mathbb{R}^n$

We have, with  $y := Ax$ ,

$$\begin{aligned}x'(\ell P)x &= x'(\Lambda' P \Lambda + M)x \\&= x'(\beta A'(D + P)A)x \\&= \beta x' A' D A x + \beta x' A' P A x \\&= \beta y' D y + \beta y' P y \geq 0\end{aligned}$$

## Application: Firm Entry and Exit

Let

- $Z$  be a finite subset of  $\mathbb{R}$
- $\Pi$  a stochastic kernel on  $Z$
- $q$  be a distribution in  $\mathcal{P}(Z)$

An individual firm's productivity  $\{z_t\}$  obeys  $\Pi$

- $z_{t+1} \sim \Pi(z_t, \cdot)$  for all  $t$

When a firm's productivity falls below  $\bar{z} \in Z$ , the firm exits

Replaced by a new firm with productivity  $z_{t+1} \sim q$

**Ex.** How does the distribution of firms (i.e., cross-section) evolve?

Solution: A randomly selected firm has

$$\mathbb{P}\{z_{t+1} = z' \mid z_t = z\} = \begin{cases} \Pi(z, z') & \text{if } z > \bar{z} \\ q(z') & \text{if } z \leq \bar{z} \end{cases}$$

The cross-sectional firm distribution sequence  $\{\psi_t\}$  satisfies

$$\begin{aligned} \psi_{t+1}(z') &= q(z') \sum_{z \leq \bar{z}} \psi_t(z) + \sum_{z > \bar{z}} \Pi(z, z') \psi_t(z) \\ &= \sum_{z \in Z} [\mathbb{1}\{z \leq \bar{z}\} q(z') + \mathbb{1}\{z > \bar{z}\} \Pi(z, z')] \psi_t(z) \end{aligned}$$



We can write

$$\psi_{t+1}(z') = \sum_{z \in Z} [\mathbb{1}\{z \leq \bar{z}\}q(z') + \mathbb{1}\{z > \bar{z}\}\Pi(z, z')] \psi_t(z)$$

as

$$\psi_{t+1}(z') = \sum_{z \in Z} Q(z, z') \psi_t(z)$$

where

$$Q(z, z') := \mathbb{1}\{z \leq \bar{z}\}q(z') + \mathbb{1}\{z > \bar{z}\}\Pi(z, z')$$

is the stochastic kernel for the “rejuvenating firm”

**Ex.**

- Under what conditions does

$$Q(z, z') := \mathbb{1}\{z \leq \bar{z}\}q(z') + \mathbb{1}\{z > \bar{z}\}\Pi(z, z')$$

have a stationary distribution?

- What's a simple condition on  $q, \Pi$  under which  $(Q, \mathcal{P}(Z))$  is globally stable?
- How would you go about computing that stationary distribution when it the condition is satisfied?

## Application: Optimal Firm Exit

An incumbent within an industry has current profits

$$\pi_t = \pi(z_t, p_t)$$

- $\{z_t\} \stackrel{\text{iid}}{\sim} \varphi$  is a firm-specific productivity process
- $\{p_t\} \stackrel{\text{iid}}{\sim} \nu$  is an exogenous price process
- $(z_t, p_t)$  takes values in  $X \subset \mathbb{R}^k$

Decision problem: Continue or exit? (optimal stopping)

Timing runs as follows

At the start of time  $t$ , observe  $z_t$  and  $p_t$

If decision = continue, then

1. receive  $\pi_t$  now
2. start next period as an incumbent

If decision = exit, then

1. receive scrap value  $s \in \mathbb{R}$
2. receive 0 in all subsequent periods

Maximizes

$$\mathbb{E} \sum_{t \geq 0} \beta^t r_t$$

Here

$$r_t = \begin{cases} \pi_t & \text{if still incumbent} \\ s & \text{upon exit} \\ 0 & \text{after exit} \end{cases}$$

Assume that  $0 < \beta < 1$  and  $\pi$  is continuous and bounded

**Ex.** Without looking at the next slide, try to write down the Bellman equation for an incumbent firm

Bellman equation:

$$v(z, p) = \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\}$$

Bellman operator:

$$Tv(z, p) = \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\}$$

**Ex.** Without looking ahead, show that  $T$  is a self-mapping on  $bcX$

If  $v \in bcX$ , then

$$\begin{aligned} |Tv(z, p)| &= \left| \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\} \right| \\ &\leq |s| + \|\pi\|_\infty + \|v\|_\infty \end{aligned}$$

Moreover,

$$\begin{aligned} Tv(z, p) &= \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\} \\ &= \max \{s, \pi(z, p) + \text{constant}\} \end{aligned}$$

is clearly continuous in  $(z, p)$

Without looking ahead, show that  $T$  is a contraction of mod  $\beta$  on  $(bcX, d_\infty)$



We use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leq |x - y| \quad (\alpha, x, y \in \mathbb{R})$$

Fix  $v, w$  in  $bcX$  and  $(z, p) \in X$

By this bound and the triangle inequality (check details),

$$\begin{aligned} |Tv(z, p) - Tw(z, p)| &\leq \beta \int |v(z', p') - w(z', p')| \varphi(dz') \nu(dp') \\ &\leq \beta \|v - w\|_\infty \end{aligned}$$

Taking the supremum over all  $(z, p) \in X$  leads to

$$\|Tv - Tw\|_\infty \leq \beta \|v - w\|_\infty$$

The value function  $v^*$  is the unique fixed point of  $T$  in  $bcX$

- proof is in lecture 14

Suppose that  $p$  and  $z$  are real valued

**Ex.** Under what conditions is  $v^*$  increasing in  $(z, p)$ ?

Hint: Look at

$$Tv(z, p) = \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\}$$

Solution: If  $\pi$  is increasing in  $(z, p)$ , then so is  $v^*$

Indeed, under this condition

$$Tv(z, p) = \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\}$$

maps  $ibcX$  into itself (increasing bounded continuous)

Moreover,  $ibcX$  is closed in  $(bcX, d_\infty)$

Hence the fixed point  $v^*$  lies in  $ibcX$

**Ex.** Can you suggest an easier way to solve the Bellman equation

$$v(z, p) = \max \left\{ s, \pi(z, p) + \beta \int v(z', p') \varphi(dz') \nu(dp') \right\}$$

Can you map it to a lower dimensional problem?

Hint: We only need to find the **expectation** of the value function

One solution: set  $h := \int v(z', p') \varphi(dz') \nu(dp')$

The Bellman equation is

$$v(z, p) = \max \{s, \pi(z, p) + \beta h\}$$

Now shift forward in time and take expectations to get

$$h = \int \max \{s, \pi(z', p') + \beta h\} \varphi(dz') \nu(dp')$$

**Ex.** Show that

$$F(h) := \int \max \{s, \pi(z', p') + \beta h\} \varphi(dz') \nu(dp')$$

is a contraction on  $\mathbb{R}_+$

Solution: The triangle inequality for integrals gives

$$|F(g) - F(h)| \leq \int |\max \{s, \pi(z', p') + \beta g\} - \max \{s, \pi(z', p') + \beta h\}| \varphi(dz') \nu(dp')$$

From the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leq |x - y| \quad (\alpha, x, y \in \mathbb{R})$$

this leads to

$$|F(g) - F(h)| \leq \int |\beta g - \beta h| \varphi(dz') \nu(dp') \leq \beta |g - h|$$

# Final comments

Exam might ask you to provide algorithms

- Describe steps of a computer solution, why the procedure works
- You don't need to write code by hand

All lemmas / thms / facts from the slides can be used freely in the exam

**Example.** “From the lectures we have  $|a \vee x - a \vee y| \leq |x - y|$ , from which it follows that ...”

Good luck :-)