

# ECON-GA 1025 Macroeconomic Theory I

## Lecture 12

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# Today's Lecture

- Optimal savings models
- Optimal growth

## Notes: Exam Prep, etc.

Exam = Monday 22nd Oct 9:30–11:30am Room 517

- Closed book
- Material covered by TJS is examinable — PS 6 provides practice
- Review lecture slides
- Updated course notes with solved exercises

Office hours: Wed 4pm–5pm

## Prequel: Topological Conjugacy

Let  $M$  and  $N$  be metric spaces

A **homeomorphism** between  $M$  and  $N$  is a continuous bijection with continuous inverse

**Example.** The map  $\tau(x) = \ln x$  from  $(0, \infty)$  to  $\mathbb{R}$  is a homeomorphism

**Example.** Let

- $M = N = \mathbb{R}^n$  with Euclidean distance
- $A$  be an  $n \times n$  matrix

When is the map  $\tau: M \rightarrow N$  defined by  $\tau(x) = Ax$  a homeomorphism?

Dynamical systems  $(M, g)$  and  $(N, h)$  are called **topologically conjugate** if  $\exists$  a homeomorphism  $\tau$  from  $M$  to  $N$  such that

$$g = \tau^{-1} \circ h \circ \tau \text{ on } M$$

Visually,

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ \tau \downarrow & & \uparrow \tau^{-1} \\ N & \xrightarrow{h} & N \end{array}$$

**Theorem.** Let  $(M, g)$  and  $(N, h)$  be topologically conjugate under homeomorphism  $\tau$

In this setting:

1.  $g^n = \tau^{-1} \circ h^n \circ \tau$  for all  $n$  in  $\mathbb{N}$
2.  $x$  is a steady state of  $(M, g)$  iff  $\tau(x)$  is a steady state of  $(N, h)$
3.  $(M, g)$  is globally stable iff  $(N, h)$  is globally stable

We say that  $(M, g)$  and  $(N, h)$  have **equivalent dynamics**

For example, let's show that

$$x \text{ fixed for } g \implies \tau(x) \text{ fixed for } h$$

Note that

$$g = \tau^{-1} \circ h \circ \tau \iff \tau \circ g = h \circ \tau$$

Now let  $x$  be a fixed point of  $g$  in  $M$

We have

$$h(\tau(x)) = \tau(g(x)) = \tau(x)$$

QED

## Prequel: Berge's Theorem of the Maximum

Let  $A$  and  $X$  be metric spaces

Let  $\Gamma$  be a nonempty compact valued correspondence from  $X$  to  $A$

- $\Gamma(x)$  is a nonempty compact subset of  $A$  for every  $x \in X$

Let  $q$  be a real valued function on

$$\mathbb{G} := \{(x, a) \in X \times A : a \in \Gamma(x)\}$$

and set

$$v(x) := \max_{a \in \Gamma(x)} q(x, a) \quad (x \in X)$$

**Theorem.** If  $\Gamma$  is continuous on  $X$  and  $q$  is continuous on  $\mathbb{G}$ , then  $v$  is well defined and continuous on  $X$



Note: We omitted the definition of continuity of correspondences

A **sufficient condition** for  $\Gamma$  to be a continuous nonempty compact valued correspondence is that  $A \subset \mathbb{R}^k$  and

$$\Gamma(x) = \{a \in A : \ell(x) \leq a \leq m(x)\}$$

where

- $\ell, m$  are continuous  $\mathbb{R}^k$  valued functions on  $X$
- $\ell(x) \leq m(x)$  for all  $x$  in  $X$

# A Generic Optimal Savings Problem

A foundation stone for

- DSGE models
- Bewley / Huggett / Aiyagari heterogeneous agent models

Agent chooses consumption path  $\{c_t\}$  to maximize

$$\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

where

- $u(c_t)$  is utility of current consumption
- $\beta$  is a discount factor satisfying  $0 < \beta < 1$

Consumption affects a **state process** via the **law of motion**

$$x_{t+1} = g(x_t, c_t, \zeta_{t+1}) \quad \{\zeta_t\} \stackrel{\text{iid}}{\sim} \varphi$$

where

- consumption  $c_t$  values in  $\mathbb{R}_+$
- the **state**  $x_t$  values in metric space  $X$
- $x_0$  is given
- the **innovation process**  $\{\zeta_t\}$  takes values in metric space  $E$

(Arbitrary metric spaces so continuous & discrete both possible)

The state restricts consumption via a **feasibility constraint**

$$c_t \in \Gamma(x_t) \subset \mathbb{R}_+$$

- for example,  $\Gamma(x) = [0, x]$  when  $x$  is assets
- $\Gamma$  is called the **feasible correspondence**

Consumption also required to be **adapted** to the history

$$\mathcal{H}_t := \{x_j\}_{j \leq t}$$

- $c_t$  cannot depend on future realizations of the state

**Assumption.** The following conditions hold:

1.  $u$  is continuous, strictly concave and strictly increasing on  $\mathbb{R}_+$
2.  $g$  is everywhere continuous
3.  $\Gamma$  is nonempty, compact valued and continuous

Collectively,  $(\beta, u, g, \varphi, \Gamma)$  called the **generic optimal savings model**

Interpretations

- Consumption and investment in a DSGE model
- Savings and asset accumulation for a household
- Optimal exploitation of a natural resource

**Example.** In Brock and Mirman (1972), a representative agent owns capital  $k_t \in \mathbb{R}_+$ , produces output

$$y_t := f(k_t, z_t)$$

Here  $f$  is the production function and  $\{z_t\}$  is an **exogenous productivity process**

Consumption is chosen to maximize  $\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$

The resource constraint is

$$0 \leq k_{t+1} + c_t \leq y_t$$

This combined with the production function leads to the law of motion

$$k_{t+1} = f(k_t, z_t) - c_t$$

The exogenous state process is assumed to follow the Markov law

$$z_{t+1} = G(z_t, \epsilon_{t+1}), \quad \{\epsilon_t\} \stackrel{\text{iid}}{\sim} \varphi$$

Maps to the generic optimal savings model  $(\beta, u, g, \varphi, \Gamma)$  if we set

- $x = (k, z)$
- law of motion

$$g((k, z), c, \xi) = \begin{pmatrix} f(k, z) - c \\ G(z, \xi) \end{pmatrix}$$

- $\Gamma(x) = [0, f(k, z)]$

What do we need for  $g$  to be continuous?

**Example.** Consider the model of household wealth dynamics

$$w_{t+1} = (1 + r_{t+1})(w_t - c_t) + y_{t+1}$$

- $w_t$  = household assets
- $c_t$  = consumption
- $y_{t+1}$  = non-financial income
- $r_{t+1}$  = the rate of return on financial assets

Assume  $y_t = y(z_t, \eta_t)$  and  $r_t = r(z_t, \zeta_t)$  where

- $z_{t+1} = G(z_t, \epsilon_{t+1})$
- $\{\eta_t\}$ ,  $\{\zeta_t\}$  and  $\{\epsilon_t\}$  are IID



Consumption is chosen to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Maps to the generic optimal savings model  $(\beta, u, g, \varphi, \Gamma)$  when

- $x = (w, z)$
- $\varphi =$  distribution of  $\tilde{\zeta} := (\epsilon, \eta, \zeta)$
- $g$  is set to

$$g((w, z), c, \tilde{\zeta}) = \begin{pmatrix} (1 + r(z, \zeta))(w - c) + y(z, \eta) \\ G(z, \epsilon) \end{pmatrix}$$

- $\Gamma((w, z)) = [0, w]$

What do we need for  $g$  to be continuous?

# Stationary Markov Policies

Recall: Consumption must be adapted to  $\mathcal{H}_t := \{x_j\}_{j \leq t}$

Means that, at each point in time  $t$ , we have

$$c_t = \sigma_t(x_0, x_1, \dots, x_t)$$

for some suitable function  $\sigma_t$  — called a **policy function**

In what follows we focus exclusively on **stationary Markov policies**

- depend only on the **current** state
- time invariant ( $\sigma_t = \sigma$ )

(In fact every optimal policy has these properties)

A stationary Markov policy is a function  $\sigma$  mapping  $X$  to  $\mathbb{R}_+$

Interpretation:

$$c_t = \sigma(x_t) \quad \text{for all } t \geq 0$$

We call  $\sigma$  a **feasible consumption policy** if

1. it is Borel measurable and
2. it satisfies

$$\sigma(x) \in \Gamma(x) \quad \text{for all } x \in X$$

Requires that

- functions nice enough to compute all expectations
- resource constraint is respected

Each  $\sigma \in \Sigma$  **closes the loop** for the state process

- determines a first order Markov process  $\{x_t\}$  via

$$x_{t+1} = g(x_t, \sigma(x_t), \zeta_{t+1})$$

This is important!

Choosing a policy  $\sigma \in \Sigma$  chooses a Markov process

Associated value is

$$v_\sigma(x) := \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(\sigma(x_t))$$

- Here  $\{x_t\}$  obeys (20) with  $x_0 = x$
- Called the  **$\sigma$ -value function**

The **value function**  $v^*$  is defined by

$$v^*(x) := \sup_{\sigma \in \Sigma} v_{\sigma}(x) \quad (x \in X)$$

A consumption policy  $\sigma^*$  is called **optimal** if it is feasible and

$$v_{\sigma^*}(x) = v^*(x) \quad \text{for all } x \in X$$

In most settings  $v^*$  satisfies the Bellman equation

$$v(x) = \max_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right\} \quad (x \in X)$$

Intuition: maximal value obtained by trading off current vs expected future rewards possible from next state

**Proposition.** Let  $(\beta, u, f, \varphi, \Gamma)$  be a generic optimal savings model  
If  $u$  is bounded, then

1.  $v^*$  is the unique solution to the Bellman equation in  $bcX$
2. A feasible consumption policy  $\sigma$  is optimal if and only if

$$\sigma(x) \in \operatorname{argmax}_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v^*(g(x, c, z)) \varphi(dz) \right\}$$

for all  $x \in X$

3. At least one such policy exists

Proof: Deferred

Consistent with earlier notation,  $\sigma \in \Sigma$  is called  **$v$ -greedy** if

$$\sigma(x) \in \operatorname{argmax}_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right\}$$

for all  $x \in X$

The last proposition states that, for  $\sigma \in \Sigma$

$$\sigma \text{ is } v^* \text{-greedy} \iff \sigma \text{ is optimal}$$

This is another version of **Bellman's principle of optimality**

We started with one optimization problem

- choosing an optimal consumption path  $c_0, c_1, \dots$  to maximize expected discounted lifetime utility

and ended up with another one

- finding a greedy policy from the value function

But we are much better off — why?



Of course, being better off is contingent on obtaining the value function

- needed to compute  $v^*$ -greedy policies

Standard method:

1. Choose initial guess  $v$
2. iterate from  $v$  via the Bellman operator

$$Tv(x) = \max_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right\}$$

**Proposition.** If  $u$  is bounded, then

1.  $T$  is a contraction of modulus  $\beta$  on  $(bcX, d_\infty)$
  2. Its unique fixed point in  $bcX$  is  $v^*$
- Why is  $u$  required to be bounded?

This assumption is not ideal, since it fails in many applications

Unbounded  $u$  issues have to be treated case-by-case

For now let's prove part 1 of the proposition

First let's show that  $T$  is a self-map on  $bcX$

Is  $Tv$  bounded on  $X$  whenever  $v \in bcX$ ?

Fix any such  $v$  and any feasible  $x$

We have

$$\begin{aligned} |Tv(x)| &\leq \max_{a \in \Gamma(x)} \left| u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right| \\ &\leq \|u\|_\infty + \beta \|v\|_\infty \end{aligned}$$

RHS does not depend on  $x$ , so  $Tv$  is bounded

Next we need to show that  $Tv$  is continuous when  $v \in bcX$

We employ **Berge's theorem of the maximum**, which tells us that  $Tv$  will be continuous whenever

$$q(x, c) := u(c) + \beta \int v(g(x, c, z))\varphi(dz)$$

is continuous on  $\mathbb{G} := \{(x, c) \in X \times \mathbb{R}_+ : c \in \Gamma(x)\}$

The tricky part is to show that

$$\int v(g(x_n, c_n, z))\varphi(dz) \rightarrow \int v(g(x, c, z))\varphi(dz)$$

when  $(x_n, c_n) \rightarrow (x, c)$

Follows from the DCT (see course notes)

Finally, let  $v$  and  $w$  be elements of  $bcX$  and fix  $x \in X$

Recalling our sup inequality

$$\left| \sup_{a \in E} f(a) - \sup_{a \in E} g(a) \right| \leq \sup_{a \in E} |f(a) - g(a)|$$

we have

$$\begin{aligned} |Tv(x) - Tw(x)| &\leq \max_{c \in \Gamma(x)} \beta \left| \int v(g(\cdot)) \varphi(dz) - \int w(g(\cdot)) \varphi(dz) \right| \\ &\leq \max_{c \in \Gamma(x)} \beta \int |v(g(x, c, z)) - w(g(x, c, z))| \varphi(dz) \end{aligned}$$

$$\therefore \|Tv - Tw\|_\infty \leq \beta \|v - w\|_\infty$$

# Problems with Analytical Solutions

For a small subset of optimal savings problems, both the optimal policy and the value function have known analytical solutions

These models are limited and simplistic!

But helpful for

- building intuition
- testing ideas
- testing numerical algorithms

Let's look at some examples

## Cake Eating with Interest

Objective function is  $\sum_{t=0}^{\infty} \beta^t u(c_t)$

Utility is

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \quad (\gamma > 0, \gamma \neq 1)$$

and

$$w_{t+1} = R(w_t - c_t)$$

Here

- $R = 1 + r$  is a gross interest rate
- $0 \leq c_t \leq w_t$  where  $w_t$  is wealth
- $\beta R^{1-\gamma} < 1$  is assumed to hold

Maps to generic savings model  $(\beta, u, g, \varphi, \Gamma)$  with

- $x_t = w_t$
- $g(x, c, \xi) = R(x - c)$
- $\Gamma(x) = [0, x]$
- $\varphi = \delta_1$

**Fact.** There exists a constant  $\theta \in (0, 1)$  such that

$$\sigma^*(w) = \theta w$$

is the optimal consumption policy

Let's verify this claim and seek the value of  $\theta$



First, observe that if  $c_t = \theta w_t$  for all  $t$ , then

$$w_t = R^t(1 - \theta)^t w \quad \text{when } w_0 = w$$

Hence

$$\begin{aligned} v^*(w) &= \sum_t \beta^t u(\theta w_t) = \sum_t \beta^t u\left(\theta R^t (1 - \theta)^t w\right) \\ &= \sum_t \beta^t \left(\theta R^t (1 - \theta)^t\right)^{1-\gamma} u(w) \\ &= \frac{\theta^{1-\gamma}}{1 - \beta (R (1 - \theta))^{1-\gamma}} u(w) \end{aligned}$$

Under the conjecture  $\sigma^*(w) = \theta w$ , the Bellman equation takes the form

$$v^*(w) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \cdot \frac{\theta^{1-\gamma}}{1-\beta(R(1-\theta))^{1-\gamma}} \cdot \frac{(R(w-c))^{1-\gamma}}{1-\gamma} \right\}$$

Taking the derivative w.r.t.  $c$  yields the first-order condition

$$c^{-\gamma} + \beta m (R(w-c))^{-\gamma} (-R) = 0$$

where

$$m := \frac{\theta^{1-\gamma}}{1-\beta(R(1-\theta))^{1-\gamma}}$$

Hence  $c^{-\gamma} = \beta m R^{1-\gamma} (w-c)^{-\gamma}$

Substituting the optimal policy  $\sigma^*(w) = \theta w$  into this equality gives us

$$(\theta w)^{-\gamma} = \frac{\beta R^{1-\gamma} \theta^{1-\gamma}}{1 - \beta (R(1-\theta))^{1-\gamma}} (1-\theta)^{-\gamma} w^{-\gamma}$$

Solving the above equality for  $\theta$  yields

$$\theta = 1 - \left( \beta R^{1-\gamma} \right)^{1/\gamma}$$

The value function becomes

$$v^*(w) = \frac{\theta^{1-\gamma}}{1 - \beta (R(1-\theta))^{1-\gamma}} u(w) = \theta^{-\gamma} u(w)$$

## Log-CD Example

Set  $u(c) = \ln c$  and

$$f(k) = Ak^\alpha, \quad 0 < A, \quad 0 < \alpha < 1$$

Let  $\{z_t\}$  be a lognormal IID sequence, with  $\ln z_t \stackrel{\mathcal{D}}{=} N(\mu, \sigma^2)$  for some  $\mu \in \mathbb{R}$  and  $\sigma > 0$

The state can be set to

$$y_{t+1} = f(y_t - c_t)z_{t+1} = A(y_t - c_t)^\alpha z_{t+1}$$

The agent maximizes

$$\mathbb{E} \sum_{t \geq 0} \beta^t \ln c_t$$

**Ex.** Conjecture that the optimal policy is linear in income  $y$

That is,  $\exists$  a positive constant  $\theta$  such that  $\sigma^*(y) = \theta y$  is optimal

Following the approach of the CRRA cake eating example

1. find the value of  $\theta$
2. obtain an expression for the value function and
3. confirm that the value function satisfies the Bellman equation

# CRRA Preferences and Stochastic Financial Returns

Let's look at a recent paper by Alexis Akira Toda (2018, JME)

He studies a heterogeneous agent economy where households optimize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta(z_t)^t u(c_t) = \mathbb{E} \sum_{t=0}^{\infty} \beta(z_t)^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

- $u$  is CRRA as before and  $\gamma > 0$
- Note that  $\beta$  is state dependent

Gives conditions for Pareto tails in the wealth distribution

Wealth dynamics are given by

$$w_{t+1} = R(z_t) (w_t - c_t)$$

The **state process**  $\{z_t\}$  is

- exogenous
- a Markov chain on finite set  $Z$  with stochastic kernel  $\Pi$

We assume that

1.  $\Pi(z, z') > 0$  for all  $z, z'$  in  $Z$
2.  $\beta(z) > 0$  and  $R(z) > 0$  for all  $z \in Z$

What does positivity of  $\Pi$  imply?

The Bellman equation is now

$$v(w, z) = \max_{0 \leq c \leq w} \left\{ u(c) + \beta(z) \sum_{z' \in Z} v[R(z)(w - c), z'] \Pi(z, z') \right\}$$

for all  $(w, z) \in X := \mathbb{R}_+ \times Z$ .

Let  $K$  be the square matrix defined by

$$K(z, z') = \beta(z) R(z)^{1-\gamma} \Pi(z, z') \quad ((z, z') \in Z \times Z)$$

In the slides below,

$$Kg(z) := \sum_{z'} g(z') K(z, z') \quad (z \in Z)$$

(Think of the matrix product with column vector  $g$ )



Toda (2018) shows that if  $r(K) < 1$ , then

1. There exists a  $g^*$  in  $\mathbb{R}^Z$  satisfying

$$g^*(z) = \left\{ 1 + [Kg^*(z)]^{1/\gamma} \right\}^\gamma \quad (z \in Z)$$

2. The optimal consumption policy is

$$\sigma^*(w, z) = g^*(z)^{-1/\gamma} w$$

3. The value function satisfies

$$v^*(w, z) = g^*(z) \frac{w^{1-\gamma}}{1-\gamma}$$

Let's

1. do the proof of part 1
2. work out how to compute the solution  $g^*$
3. study the impact of parameters

We adopt the standard pointwise partial order  $\leq$  on  $\mathbb{R}^Z$

Recall that

- self-map  $T$  on  $\mathbb{R}^Z$  is called isotone if  $g \leq h$  implies  $Tg \leq Th$
- $g \ll h$  means  $g(z) < h(z)$  for all  $z$

Let  $\psi$  be the scalar map defined by

$$\psi(t) := (1 + t^{1/\gamma})^\gamma \quad (t \geq 0)$$

Consider the operator  $S$  mapping

$$\mathcal{C} = \{g \in \mathbb{R}^Z : g \geq 0\}$$

to itself via

$$Sg(z) = \psi(Kg(z))$$

Note that, for  $g \in \mathcal{C}$ ,

$$g(z) = \left\{1 + [Kg(z)]^{1/\gamma}\right\}^\gamma, \quad \forall z \iff Sg = g$$

**Proposition.** If  $r(K) < 1$ , then  $(\mathcal{C}, S)$  is globally stable

To prove the proposition we use this result from lecture 4:

(FPT2): Let  $T$  be an isotone self-mapping on sublattice  $L$  of  $\mathbb{R}^d$  such that

1.  $\forall u \in L, \exists$  a point  $a \in L$  with  $a \leq u$  and  $Ta \gg a$
2.  $\forall u \in L, \exists$  a point  $b \in L$  with  $b \geq u$  and  $Tb \ll b$

Suppose, in addition, that  $T$  is either concave or convex

Then  $(L, T)$  is globally stable

To apply this result we need to show that

1.  $\mathcal{C}$  is a sublattice of  $\mathbb{R}^Z$
2.  $S$  is an isotone self-map on  $\mathcal{C}$
3. For all  $g \in \mathcal{C}$ ,

$$\exists \ell \in \mathcal{C} \text{ with } \ell(z) \leq g(z) \text{ and } (S\ell)(z) > \ell(z) \text{ for all } z$$

4. For all  $g \in \mathcal{C}$ ,

$$\exists m \in \mathcal{C} \text{ with } g(z) \leq m(z) \text{ and } (Sm)(z) < m(z) \text{ for all } z$$

5.  $S$  is either concave or convex

We already know that  $\mathcal{C}$  is a sublattice of  $\mathbb{R}^Z$

**Ex.** Show that  $S$  is a self-mapping on  $\mathcal{C}$

To see that  $S$  is isotone on  $\mathbb{R}^Z$ , observe that

- $S = \psi \circ K$
- the composition of two isotone maps is isotone

The map  $g \mapsto Kg$  is isotone on  $\mathbb{R}^Z$  because  $K$  is nonnegative

Indeed, if  $f \leq g$  on  $\mathbb{R}^Z$ , then

$$K(g - f)(z) = \sum_{z'} [g(z') - f(z')] K(z, z') \geq 0$$

Hence  $K(g - f) = Kg - Kf \geq 0$

Clearly  $\psi(t) = (1 + t^{1/\gamma})^\gamma$  is also isotone

**Ex.** Show that

$$\psi(t) = (1 + t^{1/\gamma})^\gamma$$

is

1. convex on  $\mathbb{R}_+$  whenever  $0 < \gamma \leq 1$
2. concave on  $\mathbb{R}_+$  whenever  $\gamma \geq 1$

**Ex.** Show that  $S = \psi \circ K$  is

1. convex on  $\mathcal{C}$  whenever  $0 < \gamma \leq 1$
2. concave on  $\mathcal{C}$  whenever  $\gamma \geq 1$

**Ex.** Show that  $S0 \gg 0$

By the **Perron–Frobenius theorem** and positivity of  $K$ ,

$$\exists e \gg 0 \text{ s.t. } Ke = r(K)e$$

- $e$  is called the **dominant eigenvector** of  $K$
- $\lambda := r(K)$  is called the **dominant eigenvalue**

**Ex.** Let  $\alpha$  be a positive constant and let  $\mathbb{1}$  be a vector of ones

Show that

$$\alpha e \gg \left( \frac{1}{1 - \lambda^{1/\gamma}} \right) \mathbb{1} \implies S(\alpha e) \ll \alpha e$$



To complete the proof we need only show that

$$\forall g \in \mathcal{C}, \exists m \geq g \text{ s.t. } Sm \ll m$$

So fix  $g \in \mathcal{C}$  and choose  $\alpha$  such that

$$\alpha e \gg \left( \frac{1}{1 - \lambda^{1/\gamma}} \right) \mathbb{1} \quad \text{and} \quad \alpha e \geq g$$

For  $m := \alpha e$ , we have  $m \geq g$  and

$$Sm = S(\alpha e) \ll \alpha e =: m$$

the proof is now done

See [toda\\_crra.ipynb](#), which solves for

- the unique positive fixed point  $g^*$  of  $S$
- the corresponding state contingent savings rate

$$s(z) := 1 - (g^*(z))^{-1/\gamma} \quad (z \in Z)$$

The simulations suggest that

1.  $\beta \leq \hat{\beta} \implies s \leq \hat{s}$
2.  $R \leq \hat{R} \implies s \leq \hat{s}$  when  $0 < \gamma < 1$
3.  $R \leq \hat{R} \implies \hat{s} \leq s$  when  $1 < \gamma < \infty$

**Ex.** Show that this is always true

## A Model with Independent Shocks

How can analysis can proceed without analytical solutions?

As a starting point, we consider a model with

- only one source of randomness — exogenous process  $\{z_t\}$
- this shock process is IID

Simplifies the problem to one with a single state variable

That state variable is  $\{y_t\}$  evolving according to

$$y_{t+1} = f(y_t - c_t)z_{t+1}$$

- **Example.** stock of a renewable resource

## Assumption.

- $f$  is continuous, concave and strictly increasing with  $f(0) = 0$
- $u$  is continuous, strictly concave and strictly increasing on  $\mathbb{R}_+$

The Bellman equation is now

$$v(y) = \max_{0 \leq c \leq y} \left\{ u(c) + \beta \int v(f(y-c)z) \varphi(dz) \right\} \quad (y \in \mathbb{R}_+)$$

The corresponding Bellman operator  $T$  is

$$Tv(y) = \max_{0 \leq c \leq y} \left\{ u(c) + \beta \int v(f(y-c)z) \varphi(dz) \right\}$$

**Theorem.**  $T$  is a contraction of modulus  $\beta$  on  $(bc\mathbb{R}_+, d_\infty)$

Moreover,

1.  $v^*$  is the unique fixed point of  $T$  in  $bc\mathbb{R}_+$
2.  $\sigma \in \Sigma$  is optimal if and only if it is  $v^*$ -greedy
3. Exactly one optimal policy and that policy is continuous

Proof:

Parts 1 and 2 follow from earlier results for the generic optimal savings model

Same for the existence component of part 3

Regarding uniqueness of the optimal policy,

**Ex.** Let  $\mathcal{C}$  be the set of increasing concave functions in  $bc\mathbb{R}_+$

- Show that  $T$  maps  $\mathcal{C}$  into itself
- Show that  $v^*$  is concave and increasing

Regarding uniqueness, observe that

$$\operatorname{argmax}_{0 \leq c \leq y} \left\{ u(c) + \beta \int v^*(f(y-c)z) \varphi(dz) \right\}$$

is a singleton

- why?
- why does this imply uniqueness of the optimal policy?

To compute  $v^*$  we can use value function iteration

Pick initial  $v_0$  in  $bc\mathbb{R}_+$  and iterate with

$$Tv(y) = \max_{0 \leq c \leq y} \left\{ u(c) + \beta \int v(f(y-c)z) \varphi(dz) \right\}$$

But how to store  $Tv$ ,  $T^2v$ , etc.?

Options:

1. Discretize the whole model
2. Use interpolation over a grid to store  $T^k v$  at each  $k$

The second option

- is less susceptible to the curse of dimensionality
- allows us to track errors

We will focus on **piecewise linear interpolation**

Advantages

- preserves monotonicity of interpolant
- preserves shape properties like concavity / convexity
- preserves contractivity of the Bellman operator

For details see the course notes



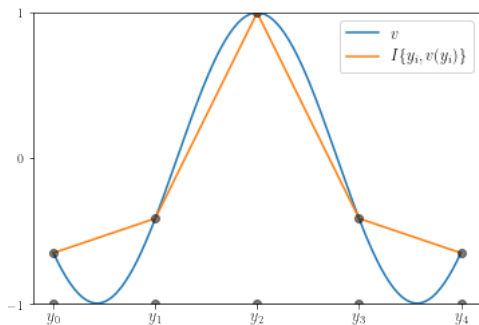


Figure: Approximation by piecewise linear interpolation

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```

draw  $\{z_j\} \stackrel{\text{iid}}{\sim} \varphi$  ;
input grid  $G_n := \{y_i\}_{i=0}^{n-1} \subset \mathbb{R}_+$  ;
input  $\{v_0(y_i)\}_{i=0}^{n-1}$ , an initial guess of  $v^*$  evaluated on  $G_n$  ;
input error tolerance  $\tau$  and set  $\epsilon \leftarrow \tau + 1$  ;
 $k \leftarrow 0$  ;
while  $\epsilon > \tau$  do
     $v_k \leftarrow I\{y_i, v_k(y_i)\}$  ;           // interpolated function
    for  $i \in \{0, \dots, n-1\}$  do
         $v_{k+1}(y_i) \leftarrow \max_{0 \leq c \leq y_i} \left\{ u(c) + \beta \frac{1}{m} \sum_{j=1}^m v_k(f(y_i - c)z_j) \right\}$  ;
    end
     $\epsilon \leftarrow \max_i |v_k(y_i) - v_{k+1}(y_i)|$  ;
     $k \leftarrow k + 1$  ;
end
return  $v_k$ 

```

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See `opt_growth.ipynb`