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Journal of Mathematical Economics 39 (2003) 135–152
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

Economic dynamical systems with multiplicative noise

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Received 24 June 2002; received in revised form 19 October 2002; accepted 21 October 2002

Abstract

The paper considers random economic systems generating nonlinear time series on the positive half-ray $\mathbb{R}_+$. Using Lyapunov techniques, new conditions for existence, uniqueness and stability of stationary equilibria are obtained. The conditions generalize earlier results from the mathematical literature, and extend to models outside the scope of existing economic methodology. Applications to growth models with productive capital are given.

JEL classification: C61; C62; O40

Keywords: Markov process; Lyapunov function

1. Introduction

Increasingly, modern economics is implemented within the framework of stochastic dynamic systems. Physical laws, equilibrium constraints and restrictions on the behavior of agents jointly determine evolution of endogenous state variable $x \in X$ according to some transition rule

$$x_{t+1} = h(x_t, z_t, \varepsilon_t), \quad t = 0, 1, \ldots, \tag{1}$$

where h is an arbitrary function, (z_t) is a sequence of exogenous forcing variables and (ε_t) is uncorrelated noise.

For some models, either z_t is constant or the endogenous variables can be conveniently redefined such that the system is autonomous:

$$x_{t+1} = h(x_t, \varepsilon_t), \quad t = 0, 1, \ldots \tag{2}$$

Assume that this is the case.

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doi:10.1016/S0304-4068(03)00006-5

Of primary concern is whether the autonomous system (2) is in some sense stationary, in which case one can anticipate convergence of the sequence of distributions (φ_t) associated with the sequence of random variables (x_t) to some unique limiting distribution φ^*. The latter is then interpreted as the long-run equilibrium of the economy (2). Typically, comparative dynamics (policy simulation) will be performed by analyzing the relationship between its moments and the underlying structural parameters contained in the function h and the distribution of the shock ε.

When h is linear on real vector space, (2) is the standard autoregression (AR) model. Conditions for stationarity are familiar from elementary time series analysis (Hamilton, 1994). When the map is nonlinear, dynamic behavior is potentially more complicated. General conditions for existence of unique and stable equilibria are not known.

In this case, a common approach is to linearize (2) using a first-order Taylor expansion or similar technique, and then examine the stability properties of the resulting AR model. However, it is by no means clear that stability properties obtained for the AR model have *any* homeomorphic implications for the behavior of the true model (2). In other words, it is not in general legitimate to infer stability of (2) from stability of the corresponding linear form. Moreover, linearization may eliminate important features of the model.[1]

A more correct method is to examine the Markov chain generated by (2), and determine whether appropriate conditions for stability of Markovian systems are satisfied. An early example is Brock and Mirman (1972). An excellent survey of sufficient conditions is provided by Futia (1982). Stokey et al. (1989, chapter 13) outline ways to verify these and related conditions for common economic models. Hopenhayn and Prescott (1992) develop new sufficient conditions using only monotonicity and a mixing condition. Bhattacharya and Majumdar (2001) obtain exponential convergence in the Kolmogorov metric for real-valued systems that satisfy a "splitting" condition.

In this paper we focus on a specific class of models that arise naturally in economics. In particular, we assume that the shock ε is multiplicative, and that the state space for the endogenous variable x_t is the positive half-ray $\mathbb{R}_+ = [0, \infty)$. That is,

$$x_{t+1} = g(x_t)\varepsilon_t, \quad t = 0, 1, \dots, \tag{3}$$

where g: $\mathbb{R}_+ \to \mathbb{R}_+$, and $\varepsilon_t \in \mathbb{R}_+$. The importance of these models within economics stems from inherently non-negative state variables, such as prices or physical quantities. A case in point is stochastic growth theory (Stokey et al., 1989, Section 13.3), which in turn provides foundations for the real business cycle and other macroeconomic literature.

While (3) excludes a larger model architecture than previous studies, it is demonstrated that the additional structure can be exploited to obtain results that have considerable generality within this class. For example, stability results are obtained for models that may satisfy *none* of the sufficient conditions used in the well-known framework of Stokey et al. (1989, Section 12.4). Further, the approach leads naturally to sufficient conditions stated

[1] For example, Durlauf and Quah (1999) found evidence to the effect that standard linearization procedures applied to Solow–Ramsey growth models fail to extract nonlinear local increasing returns dynamics that are critical to understanding the evolution of the cross-country income distribution.

directly in terms of the primitives g and ε; such conditions are easy to verify in applications. Third, the temptation to compactify the state space is resisted, permitting incorporation of standard econometric shocks. Fourth, equilibria are realized as fixed points of a contractive linear operator, and are therefore amenable to approximation by numerical methods.

The stability of (3) has previously been studied in the mathematical literature. In particular, there exists a well-known set of sufficient conditions due to Horbacz (1989, Theorem 1). The results obtained here provide a general principle which yields the conditions of Horbacz as a special case.[2]

Our arguments are based on the framework for studying integral Markov semigroups in the function space L_1 proposed by Lasota (1994). Previously, Stachurski (2002) has applied Lasota's method to the stochastic neoclassical growth problem.

The paper proceeds as follows. Section 2 provides background on the formal structure. Section 3 states our results. Section 4 gives applications. Section 5 gives proofs.

2. Formulation of the problem

In this section, a more formal treatment of the model (3) is given. To begin, let $\mathbb{R}$ be the real numbers, let $\mathcal{B}$ be the Borel sets of $\mathbb{R}$, let $\mathbb{R}_+ = [0, \infty)$, and let $\mathcal{B}_+ = \mathcal{B} \cap \mathbb{R}_+$. The Lebesgue measure is denoted by μ. Integration where the measure is not made explicit is taken with respect to μ; integration using the symbol $\int$ without subscript is taken over $\mathbb{R}_+$.

Let $\mathcal{M}$ be the vector lattice of finite signed measures on $(\mathbb{R}_+, \mathcal{B}_+)$ with total variation norm. Let $\mathcal{P}$ be the elements $\nu \in \mathcal{M}$ such that $\nu \geq 0$ and $\nu(\mathbb{R}_+) = \|\nu\| = 1$. The subset $\mathcal{P}$ will be called the *distributions* on $\mathbb{R}_+$.

Further, let $L_1(\mu)$ be the space of μ-integrable real functions on the measurable space $(\mathbb{R}_+, \mathcal{B}_+)$. As usual, $L_1(\mu)$ is interpreted as a Banach lattice of equivalence classes; functions equal off a μ-null set are identified. The sets $\mathcal{M}$ and $L_1(\mu)$ are related in that $L_1(\mu)$ is isometrically and lattice isomorphic to a subset of $\mathcal{M}$ under Radon–Nikodým (RN) differentiation with respect to μ.

A *density function* on $\mathbb{R}_+$ is an element $\varphi \in L_1(\mu)$ such that $\varphi \geq 0$ and $\int \varphi = \|\varphi\| = 1$. The set of all density functions is denoted $D(\mu)$.

In the model, random outcomes are implemented as follows. For some measurable space $(\Omega, \mathcal{F})$, and for some fixed probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, a state of nature is selected from Ω according to $\mathbb{P}$, and mapped into $\mathbb{R}_+$ by random variable $\varepsilon \colon \Omega \to \mathbb{R}_+$. The random variable defines a distribution $\Psi \in \mathcal{P}$ associating event $B \in \mathcal{B}_+$ with the real number $\mathbb{P}[\varepsilon^{-1}(B)] \in [0, 1]$.

Two basic assumptions on the structure of (3) are required. First,

Assumption 1. The shocks (ε_t) are uncorrelated and identically distributed by density function ψ on $\mathbb{R}_+$.

Assumption 2. The map g is strictly positive almost everywhere on $\mathbb{R}_+$.[3]

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[2] A proof of this statement is available from the author.

[3] Thus, we accommodate the possibility that g may be zero at a finite number of points.

Assumption 1 states that Ψ can be represented by a density function. In other words, there exists a unique density $\psi \in D(\mu)$ satisfying $\int_B \psi = \Psi(B)$ for all $B \in \mathcal{B}_+$; ψ is the Radon–Nikodým (RN) derivative of Ψ with respect to μ.

Definition 1. Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{B}_+$-measurable function. In what follows, a perturbed dynamical system on $\mathbb{R}_+$ is defined by a pair (g, ψ), where, given current state value $x_t \in \mathbb{R}_+$, a shock $\varepsilon_t \in \mathbb{R}_+$ is selected independently from density $\psi \in D(\mu)$, and the next period state is realized as in (3).

2.1. The standard model

Dynamics of perturbed dynamical systems are usually described in terms of transition kernels (Futia, 1982, Definition 1.1). Let $\mathbf{1}_B: \mathbb{R}_+ \to \{0, 1\}$ be the characteristic function for $B \in \mathcal{B}_+$. The pair (g, ψ) determines a Markov process on $\mathbb{R}_+$ with transition kernel $\boldsymbol{N}$,

$$\boldsymbol{N} : \mathbb{R}_+ \times \mathcal{B}_+ \ni (x, B) \mapsto \int \mathbf{1}_B[g(x)z]\psi(z)\,\mathrm{d}z \in [0, 1].$$

The value $\boldsymbol{N}(x, B)$ should be interpreted as the conditional probability that the next period state is in Borel set B, given that the current state is equal to x. A Markov process is fully characterized by its transition kernel.

We seek to derive using $\boldsymbol{N}$ a recursion that links successive marginal distributions of the state variables. Let B be any Borel set, and let $\nu_t \in \mathcal{P}$ be the marginal distribution for the random variable x_t.[4] By the law of total probability, if ν_{t+1} is the distribution for x_{t+1}, then

$$\nu_{t+1}(B) = \int \boldsymbol{N}(x, B)\nu_t\,(\mathrm{d}x). \tag{4}$$

Intuitively, the probability that the state variable is in B next period is the sum of the probabilities that it travels to B from x across all $x \in \mathbb{R}_+$, weighted by the probability $\nu_t(dx)$ that x occurs as the current state.

Following Futia (1982), Stokey et al. (1989), and other authors, the relationship (4) is redefined in terms of operators. Suppose we define an operator $P: \mathcal{M} \ni \nu \mapsto P\nu \in \mathcal{M}$ by

$$P\nu(B) = \int \boldsymbol{N}(x, B)\nu\,(\mathrm{d}x). \tag{5}$$

It follows from (4) and (5) that if ν_t is the distribution for the current state x_t, then $P\nu_t$ is the distribution for the next period state x_{t+1}.

Evidently, $P\mathcal{P} \subset \mathcal{P}$. A linear self-mapping on $\mathcal{M}$ satisfying $P\mathcal{P} \subset \mathcal{P}$ is called a *Markov operator*.[5]

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[4] The distribution for the entire stochastic process $(x_t)_{t\geq 0}$ can be constructed uniquely from the transition kernel and an initial value x_0 (Shiryaev, 1996, Theorem II.9.2). The real number $\nu_t(B)$ is the probability that this distribution assigns to the event $x_t \in B$ and $x_s \in \mathbb{R}_+$ for all other $s \neq t$.

[5] The operator P corresponds to, for example, T^* in Futia (1982, p. 380). Markov operators are called *stochastic operators* by some authors. Our terminology follows the literature on Markov processes in L_1.

Repeated iteration of P on a fixed distribution ν is equivalent to moving forward in time. If P^t is defined inductively by $P^t = PP^{t-1}$ and $P^1 = P$, and if ν is the current marginal distribution for the state variable, then $P^t\nu$ is the distribution t periods hence.

2.2. The L_1 method

The above framework essentially follows Futia (1982), Stokey et al. (1989) and subsequent authors. However, in this paper we diverge slightly, approaching Markov chains generated by (g, ψ) using the L_1 method (Hopf, 1954); stochastic processes are studied by analyzing evolution of density functions, which represent the marginal distributions of current and future state variables. The advantage is that we can exploit a very useful technique for studying Markov chains in L_1 due to Lasota (1994).

Embedding the Markov problem in L_1 requires that the transition probabilities can be represented by density functions. This was the purpose of Assumption 2. It can be verified under this assumption that for almost all x, the distribution $B \mapsto \boldsymbol{N}(x, B)$ is absolutely continuous with respect to μ, and can therefore be represented by density $y \mapsto p(x, y)$. Heuristically, the number $p(x, y)\,\mathrm{d}y$ is the probability of traveling from state x to state y in one step. In this paper, p is called the *density kernel* corresponding to (g, ψ).

For x such that $g(x) > 0$,

$$p(x, y) = \psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}, \tag{6}$$

because changing variables shows that for any $B \in \mathcal{B}_+$ and any such x,

$$\int_B p(x, y)\,\mathrm{d}y = \int \mathbf{1}_B[g(x)z]\psi(z)\,\mathrm{d}z = \boldsymbol{N}(x, B).$$

Hence, p represents $\boldsymbol{N}$ as claimed. For other x set $p(x, \cdot)$ equal to any density.[6]

Using p, the Markov operator P corresponding to (g, ψ) can now be reinterpreted as a linear self-mapping on the function space $L_1(\mu)$. Specifically, if $h \in L_1(\mu)$, then

$$Ph(y) = \int p(x, y)h(x)\,\mathrm{d}x. \tag{7}$$

It can be verified that the two definitions (5) and (7) of P are equivalent for the absolutely continuous measures in $\mathcal{M}$ when these measures and their RN derivatives in $L_1(\mu)$ are identified.[7]

Note that $PD(\mu) \subset D(\mu)$, as can be shown directly using Fubini's theorem. As before, if φ is the current marginal density for the state variable, then $P^t\varphi$ is that of the state t periods hence.

Definition 2. Let (g, ψ) be a perturbed dynamical system satisfying Assumptions 1 and 2. Let P be the corresponding Markov operator. An equilibrium or steady state for (g, ψ) is

[6] Density kernels need be defined only up to the complement of a null set—systems with kernels equal $\mu \times \mu$-a.e. have identical dynamics and we do not distinguish between them in what follows.

[7] For further discussion see Lasota and Mackey (1994, chapter 12).

a density φ^* on $\mathbb{R}_+$ such that $P\varphi^* = \varphi^*$. An equilibrium φ^* is called unique if there exists no other fixed point of P in the space $D(\mu)$, and globally stable if $P^t\varphi \to \varphi^*$ in the $L_1(\mu)$ metric as $t \to \infty$ for every $\varphi \in D(\mu)$.

This equilibrium concept is standard (Stokey et al., 1989, pp. 317–318). Note that stability is defined in terms of the norm topology on $L_1(\mu)$. Thus, distributions corresponding to the density functions in Definition 2 converge in the strong (total variation) topology on $\mathcal{M}$. Existing techniques typically obtain only weak or weak-star stability.[8]

3. Results

In this section, the main results are stated. Models are required to satisfy some combination of the following four conditions.

The first uses the notion of a *Lyapunov function on* $\mathbb{R}_+$, which we define to be a continuous, non-negative function V from $\mathbb{R}_+$ into $\mathbb{R}_+ \cup \{\infty\}$ such that $V(0) = \infty$, $V(x) < \infty$ for $x > 0$ and $\lim_{x\to\infty} V(x) = \infty$.

Condition 1. Corresponding to (g, ψ), there exists a Lyapunov function V on $\mathbb{R}_+$ and constants $\alpha, C \geq 0$, $\alpha < 1$, such that

$$\int V[g(x)z]\psi(z)\,\mathrm{d}z \leq \alpha V(x) + C, \quad \forall x \in \mathbb{R}_+.$$

The function V in Condition 1 is large at 0 and $+\infty$. The condition should be interpreted as a restriction on the probability that the state variable moves toward these limits without bound.

Condition 2. The density ψ is strictly positive almost everywhere on $\mathbb{R}_+$.[9]

Most "named" densities on $\mathbb{R}_+$ have this property, such as the lognormal, exponential, χ-squared, gamma, and Weibull densities.

Condition 3. For some $M < \infty$, ψ satisfies $\psi(z)z \leq M$, $\forall z \in \mathbb{R}_+$.

Condition 3 also holds for the lognormal, exponential, χ-squared, gamma and Weibull distributions. The condition is used here to bound the probability that ψ assigns to closed intervals in $\mathbb{R}_+ \setminus \{0\}$.

Theorem 3. *Let (g, ψ) be an economy on $\mathbb{R}_+$ satisfying Assumptions 1 and 2. If g and ψ also satisfy Conditions 1–3, then (g, ψ) has a unique, globally stable equilibrium.*

[8] Using coarser topologies is not a free lunch. For example, in every infinite dimensional normed vector space U, there exists a net of points all with norm one converging to the zero element in the weak topology induced by the norm dual of U.

[9] When this is the case, the same distribution for ε can be represented by a density that is positive *everywhere* on $\mathbb{R}_+$. Hence, we can assume without loss of generality that $\psi(z) > 0$, $\forall z \in \mathbb{R}_+$.

Alternatively, suppose that

Condition 4. The map g is weakly monotone increasing on the nonempty interval $[0, r)$, and $g(x) \geq b > 0$ on $[r, \infty)$.

Theorem 4. *Let (g, ψ) be an economy on $\mathbb{R}_+$ satisfying Assumptions 1 and 2. If g and ψ also satisfy Conditions 1, 2 and 4, then (g, ψ) has a unique, globally stable equilibrium.*

The proofs of Theorems 3 and 4 are given in Section 5.

Corollary 5. *Let (g, ψ) be an economy on $\mathbb{R}_+$ satisfying Assumptions 1 and 2. If g is weakly monotone increasing and, in addition, g and ψ together satisfy Conditions 1 and 2, then (g, ψ) has a unique, globally stable equilibrium.*

Proof. Evidently, Condition 4 is also satisfied if Assumption 2 holds and g is weakly monotone increasing on $\mathbb{R}_+$. Theorem 4 then implies the stated result. □

4. Applications

We analyze the dynamics of two capital accumulation models using these results. One is a standard overlapping generations model, while the other is of growth with state-dependent shocks.

4.1. Overlapping generations

In the deterministic case, dynamics of the overlapping generations model with productive capital were extensively studied by Galor and Ryder (1989). They establish convergence to a unique, nontrivial equilibrium under a strengthened Inada condition. Here, the same result is extended to the stochastic case.[10]

The framework is as follows. Agents live for two periods, working in the first and livingoff savings in the second. Savings in the first period forms capital stock, which in the following period will be combined with the labor of a new generation of young agents for production under the technology $y_t = F(k_t, \ell_t)\varepsilon_t$. Here, y_t is income, k_t is capital and ℓ_t is the number of young agents, all of whom supply inelastically one unit of labor. For convenience we assume that population is constant ($\ell_t = \ell$), and set $f(k) = F(k, \ell)$. Following Galor and Ryder (1989, p. 362), we assume that $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ has the usual properties $f(0) = 0$, $f \in C^2$, $f' > 0$, $f'' < 0$, and

$$\lim_{k\downarrow 0} f'(k) = \infty, \qquad \lim_{k\uparrow\infty} f'(k) = 0.$$

[10] Previously, Wang (1993) studied the stochastic overlapping generations model under the assumption that productivity shocks have compact support.

In addition, Galor and Ryder (1989, Proposition 5 and Corollary 1) introduce the extended Inada condition

$$\lim_{k\downarrow 0}[-kf''(k)] > 1.$$

The shocks (ε_t) are uncorrelated and identically distributed on $\mathbb{R}_+$ according to density ψ. We assume that ψ is strictly positive on $\mathbb{R}_+$.

As Galor and Ryder point out (1989, Lemma 1, p. 365), restrictions on the utility function are necessary to obtain unique self-fulfilling expectations. Here, we assume that young agents maximize utility

$$U(c_t, c'_{t+1}) = \ln c_t + \beta\mathbb{E}(\ln c'_{t+1}), \quad \beta \in (0, 1), \tag{8}$$

subject to the budget constraint

$$s_t = (w_t - c_t), \quad c'_{t+1} = s_t R_{t+1},$$

where s is savings from wage income, c (c', respectively) is consumption while young (old, respectively), w is the wage rate and R is the gross rate of return on savings. Competitive markets imply that firms pay inputs their marginal factor product. That is,

$$R_t(k_t, \varepsilon_t) = f'(k_t)\varepsilon_t, \qquad w_t(k_t, \varepsilon_t) = [f(k_t) - k_t f'(k_t)]\varepsilon_t. \tag{9}$$

Thus, at time t, households choose s_t to maximize

$$\ln (w_t(k_t, \varepsilon_t) - s_t) + \beta\mathbb{E}\ln [s_t R_{t+1}(k_{t+1}, \varepsilon_{t+1})], \tag{10}$$

using their knowledge of the distribution ψ of ε_t to evaluate the expectations operator, as well as their current belief that next period capital stock will be k_{t+1}. In self-fulfilling expectations equilibrium their beliefs are realized, with

$$k_{t+1} = s_t = \frac{\beta}{1+\beta} h(k_t)\varepsilon_t, \tag{11}$$

where $h(k) = [f(k) - kf'(k)]$. We have set $\ell = 1$ for convenience.

Proposition 6. *Assume that the Galor–Ryder condition* (*GR*) *holds. If, in addition,* $\mathbb{E}(\varepsilon) < \infty$ and $\mathbb{E}(1/\varepsilon) < \beta/(1+\beta)$, *then the economy* (11) *has a unique and globally stable stochastic equilibrium.*

Remark 1. The bound $\mathbb{E}(1/\varepsilon) < \beta/(1+\beta)$ is needed to restrict weight in the left-hand tail of ψ, preventing the economy from collapsing to zero as a result of adverse shocks.

Proof. We verify that (11) satisfies the conditions of Corollary 5. To this end, let $D = \beta/(1+\beta)$, and let $g(k) = Dh(k)$. It follows from our assumptions on f that the function $k \mapsto h(k)$ is zero at zero, continuously differentiable, strictly increasing and satisfies $\lim_{k\downarrow 0} h'(k) > 1$. This last fact—which is equivalent to (GR)—implies (via the mean value theorem) that

$$\exists\, \delta > 0 \quad \text{such that} \quad h(k) \geq k, \quad \forall k \in [0, \delta). \tag{12}$$

Evidently, Assumptions 1 and 2 are satisfied. Regarding Condition 1, consider the Lyapunov function defined by $V(k) = (1/k) + k$. We have

$$\int V[g(k)z]\psi(z)\,\mathrm{d}z = \mathbb{E}\left(\frac{1}{\varepsilon}\right)\frac{1}{g(k)} + \mathbb{E}(\varepsilon)g(k). \tag{13}$$

Consider the first term in the right-hand side of (13). By (12),

$$\mathbb{E}\left(\frac{1}{\varepsilon}\right)\frac{1}{g(k)} \le \alpha_1 \frac{1}{k}, \quad \forall k \in [0, \delta), \tag{14}$$

where $\alpha_1 = \mathbb{E}(1/\varepsilon)D^{-1} < 1$. In addition, monotonicity of g yields

$$\mathbb{E}\left(\frac{1}{\varepsilon}\right)\frac{1}{g(k)} \le \mathbb{E}\left(\frac{1}{\varepsilon}\right)\frac{1}{g(\delta)}, \quad \forall k \in [\delta, \infty). \tag{15}$$

Combining (14) and (15) gives

$$\mathbb{E}\left(\frac{1}{\varepsilon}\right)\frac{1}{g(k)} \le \alpha_1 \frac{1}{k} + C_1, \quad \forall k \in \mathbb{R}_+, \tag{16}$$

where C_1 is a finite constant. □

Consider now the second term in the right-hand side of (13). By the assumptions on f it is clear that the function $k \mapsto \mathbb{E}(\varepsilon)Df(k)$ can be majorized on $\mathbb{R}_+$ by an affine function $k \mapsto \alpha_2 k + C_2$, where α_2 and C_2 are non-negative constants, $\alpha_2 < 1$. That is,

$$\mathbb{E}(\varepsilon)g(k) \le \mathbb{E}(\varepsilon)Df(k) \le \alpha_2 k + C_2, \quad \forall k \in \mathbb{R}_+. \tag{17}$$

Let $\alpha = \max(\alpha_1, \alpha_2)$, and let $C = C_1 + C_2$. Substituting (16) and (17) into (13) gives

$$\int V[g(k)z]\psi(z)\,\mathrm{d}z \le \alpha\left(\frac{1}{k} + k\right) + C = \alpha V(k) + C. \tag{18}$$

Since $\alpha < 1$, Condition 1 is satisfied.

In addition, Condition 2 is satisfied by hypothesis, and $k \mapsto g(k)$ is monotone increasing on $\mathbb{R}_+$. Thus, all of the conditions of Corollary 5 are verified.

4.2. *Growth with state-dependent shocks*

Recently, several authors have argued that the probability of adverse shocks decreases as economies develop (Acemoglu and Zilibotti, 1997; Cetorelli, 2002). These models have interesting implications for income dynamics. In particular, they may help explain the emergence of bimodality in the cross-country income distribution (Cetorelli, 2002).

To date, the analysis has focused on the case where shocks are discrete—either by assumption or construction. Discrete shocks are simple to analyze mathematically, but inhibit econometric implementation, the theory of which is usually cast in the standard statistical framework of continuous shocks defined by density functions. In this section, we analyze the dynamics of a model with continuous, state-dependent shocks.

Consider again the model of the previous section, this time with the following modifications. The production function is assumed to be Cobb–Douglas. That is,

$$y_t = k_t^\alpha \ell_t^{1-\alpha} \varepsilon_t, \quad 0 < \alpha < 1.$$

Also, shocks are not identically distributed. Instead, the probability of adverse shocks decreases in the aggregate stock of capital. Formally, if F_k is the cumulative distribution of ε when $k_t = k$, then, $\forall x \in \mathbb{R}_+$,

$$F_{k'}(x) \le F_k(x) \quad \text{whenever} \quad k' \ge k. \tag{19}$$

A convenient way to represent this situation is to set

$$F_k(x) = \int_0^{x/\gamma(k)} \psi(z)\, \mathrm{d}z, \tag{20}$$

where ψ is a fixed density function on $\mathbb{R}_+$, and γ is some strictly positive function. If γ is nondecreasing, then (19) holds.

The model can be solved along similar lines as the previous section. Firms do not take into account the external affect of aggregate capital deepening on the likelihood of adverse shocks, and again offer wage and interest rates as in (9). Young agents again fix a belief for next period capital stock k_{t+1} and solve (10), using (20) to evaluate expectations. In equilibrium, $k_{t+1} = s_t$ (normalize $\ell = 1$), yielding the law of motion

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha \varepsilon_t. \tag{21}$$

Dynamic analysis is considerably complicated by the dependence of probabilities on the state variable. Nevertheless,

Proposition 7. *If the function γ satisfies $0 < \gamma' \le \gamma(k) \le \gamma'' < \infty$ for all k, then for any ψ from a class of distributions that includes the lognormal distributions, the economy (21) has a unique, globally stable stochastic equilibrium.*

Proof. Since the sequence of shocks (ε_t) is not identically distributed, Theorems 3 and 4 do not immediately apply. Note, however, that the influence of k on the shock can be decoupled as follows. If η is a random variable on $\mathbb{R}_+$ distributed according to ψ, then $\eta_t\gamma(k_t) = \varepsilon_t$, as can be deduced from (20). Setting $g(k) = (\beta/(1+\beta))(1-\alpha)k^\alpha\gamma(k)$, we have

$$k_{t+1} = g(k_t)\eta_t, \tag{22}$$

which is in the form of (3). We verify that (22) satisfies the conditions of Theorem 3. By construction, Assumption 1 holds, as does Assumption 2. By the hypotheses of the proposition, we may assume that ψ satisfies Conditions 2 and 3, and that $\mathbb{E}\,|\ln\eta| < \infty$. Regarding Condition 1, let $V(k) = |\ln k|$. The function V so constructed is a Lyapunov function on $\mathbb{R}_+$. Moreover,

$$\begin{aligned}\int V[g(k)z]\psi(z)\,\mathrm{d}z &= \int |\ln D + \alpha\ln k + \ln\gamma(k) + \ln z|\psi(z)\,\mathrm{d}z \le \alpha|\ln k| + C \\ &= \alpha V(k) + C,\end{aligned}$$

where D is a positive constant, $C = |\ln D| + \sup_k |\ln \gamma(k)| + \mathbb{E}\,|\ln \eta|$. Since $\alpha < 1$ and $C < \infty$, Condition 1 also holds, completing the proof. □

5. Proofs

Verification of Theorems 3 and 4 proceeds by outlining a framework for obtaining existence, uniqueness and stability of equilibria, and then establishing the required lemmas. The framework for studying integral Markov operators used here is due to Lasota (1994). Our exposition of Lasota's method draws on Stachurski (2002).

By the definition of equilibrium, the proof requires a fixed point argument for a mapping $T: U \to U$ on a metric space (U, ρ), where in the present case T corresponds to the Markov operator P defined in (7), U is the space of density functions $D(\mu)$, and ρ is the distance in $D(\mu)$ induced by the L_1 norm.

A standard result which gives existence, uniqueness and stability of equilibrium in the form desired here is the Banach contraction theorem. However, the contraction condition of Banach is not always satisfied under Conditions 1–4. Here, we pursue an alternative contraction-based argument, using a slightly weaker condition.

Definition 8. Let U be a metric space, and let $T: U \to U$. The map T is called *contracting* on U if

$$\rho(Tx, Tx') < \rho(x, x'), \quad \forall x, x' \in U, \quad x \neq x'. \tag{23}$$

Remark 2. Contracting maps have at most one fixed point: If x and x' are any two fixed points of T in U, then $\varrho(Tx, Tx') = \varrho(x, x')$, and hence $x = x'$ by (23).

Lemma 9. *Let (g, ψ) be a perturbed dynamical system satisfying Assumptions 1 and 2. If Condition 2 holds, then the associated Markov operator P is contracting on $D(\mu)$ with respect to the metric induced by the $L_1(\mu)$ norm.*

By Remark 2, Lemma 9 establishes the uniqueness component of Theorems 3 and 4. The result of Lemma 9 is already known. For completeness, a short proof is given in Appendix A.

Consider the remaining problems of existence and stability. It is known that when $T: U \to U$ is contracting on a *compact* metric space (U, ϱ), then T has a unique fixed point $x^* \in U$.[11] Uniqueness is by Remark 2. To prove existence, define $r: U \to \mathbb{R}$ by $r(x) = \varrho(Tx, x)$. Evidently, r is continuous. Since U is compact, r has a minimizer x^*. But then $Tx^* = x^*$ must hold, because otherwise

$$r(Tx^*) = \varrho(TTx^*, Tx^*) < \varrho(Tx^*, x^*) = r(x^*),$$

contradicting the definition of x^*.

[11] Strictness of the inequality in (23) is necessary for both uniqueness and existence. For example, existence fails if U is the boundary of the unit sphere in $\mathbb{R}^2$, and $Tx = -x$.

It is less well known but also true that $T^n x \to x^*$ as $n \to \infty$ for all $x \in U$. To see this, pick any $x \in U$. Consider the real sequence $\varrho(T^n x, x^*)$. Since T is contracting, the sequence is monotone decreasing, and therefore has limit $\alpha \geq 0$. It is clear that every limit point $x' \in U$ of the trajectory $(T^n x)$ satisfies $\varrho(x', x^*) = \alpha$. By compactness of U, the trajectory has at least one limit point $x' \in U$ $(\exists T^{n(k)} x \to x')$. As T is a contraction and therefore continuous, Tx' must also be a limit point $(T^{n(k)+1} x \to Tx')$. If $\alpha = 0$, then we are done. Suppose otherwise. Then x' and x^* are distinct, in which case

$$\alpha = \varrho(x', x^*) > \varrho(Tx', Tx^*) = \varrho(Tx', x^*) = \alpha.$$

Contradiction.

We have proved that contractiveness of the operator and compactness of the space together imply existence, uniqueness and global stability of equilibrium. In the case of the perturbed dynamical system (g, ψ), while P is contracting on the metric space $D(\mu)$ with L_1 distance by Lemma 9, $D(\mu)$ is *not* compact in the L_1 topology. Some weakening of the compactness condition is required. Consider the following approach. Suppose that, in addition to contractiveness of P on $D(\mu)$, the set of iterates $\{P^t \varphi : t \geq 0\}$ is precompact (i.e. has compact closure) for any initial distribution $\varphi \in D(\mu)$.[12] Such a property is called *Lagrange stability*. Let $\Gamma(\varphi)$ denote the closure of $\{P^t \varphi : t \geq 0\}$. It is straightforward to check that $P\Gamma(\varphi) \subset \Gamma(\varphi)$.[13] In this case, P is a contracting self-mapping on the compact set $\Gamma(\varphi)$. By the prior discussion, P has a fixed point φ^* in $\Gamma(\varphi) \subset D(\mu)$, and $P^t \varphi \to \varphi^*$ in L_1 norm. Finally, since P is a contraction on the whole space $D(\mu)$, the fixed point φ^* is unique and does not depend on φ.

Thus, it remains only to establish Lagrange stability of the Markov operator P associated with (g, ψ) on the density space $D(\mu)$. Lasota (1994) has made the important insight that in the case of integral Markov operators such as (7), it is sufficient to prove that $\{P^t \varphi : t \geq 0\}$ is *weakly* precompact for every $\varphi \in D(\mu)$. The reason is that integral Markov operators map weakly precompact subsets of $L_1(\mu)$ into strongly precompact subsets.[14] Therefore, if $\{P^t \varphi : t \geq 0\}$ is weakly precompact, then $\{P^t \varphi : t \geq 1\}$ is strongly precompact. But then $\{P^t \varphi : t \geq 0\}$ is also strongly precompact.

In fact, Lasota (1994, Proposition 3.4) has used a Cantor diagonal argument to show that weak precompactness of $\{P^t \varphi : t \geq 0\}$ need only be established for a collection of φ such that the norm-closure of the collection contains $D(\mu)$. In summary, then, both Theorems 3 and 4 will be verified if we are able to show that under the hypotheses of either theorem there exists a set $\mathcal{D}$ such that $\mathcal{D}$ is dense in $D(\mu)$ and $\{P^t \varphi : t \geq 0\}$ is weakly precompact for each $\varphi \in \mathcal{D}$:

Proposition 10. *Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 1 and 2, and let P be the associated Markov operator. If Condition 1 and either one of Condition 3* or 4 *holds, then there exists a set $\mathcal{D} \subset L_1(\mu)$ such that $\mathcal{D}$ is norm-dense in $D(\mu)$ and $\{P^t \varphi : t \geq 0\}$ is weakly precompact for each $\varphi \in \mathcal{D}$.*

[12] We use the convention that P^0 is the identity map.

[13] Note that P is a contraction and therefore continuous.

[14] For a proof see, for example, Lasota (1994, Theorem 4.1).

Proof. The proof is broken down into a series of lemmata. Lemma 11 exhibits a suitable set $\mathcal{D}$ that is dense in $D(\mu)$. Lemma 12 gives a general condition for weak precompactness in L_1. The condition consists of parts (i) and (ii). Lemma 13 establishes part (i) for the set $\{P^t\varphi\colon t \geq 0\}$ when $\varphi \in \mathcal{D}$. Lemma 14 establishes part (ii) for the same collection. This completes the proof of the proposition. □

Lemma 11. *Let the conditions of Proposition 10 hold. Let V be the Lyapunov function in Condition 1. Let $\mathcal{D}$ be the set of all φ in $D(\mu)$ such that*

$$\int V(x)\varphi(x)\,\mathrm{d}x < \infty. \tag{24}$$

The set $\mathcal{D}$ is norm-dense in $D(\mu)$.

Proof. The function V is bounded on compact subsets of $\mathbb{R}_+ \setminus \{0\}$ by continuity. The set of densities with compact support in $\mathbb{R}_+ \setminus \{0\}$ therefore resides in $\mathcal{D}$, and is easily seen to be norm-dense in $D(\mu)$. □

Lemma 12. *A collection $\{\varphi_\lambda\}$ in $D(\mu)$ is weakly precompact whenever*

(i) $\forall\varepsilon > 0, \exists\,\delta > 0$ *such that*

$$A \in \mathcal{B}_+ \quad \textit{and} \quad \mu(A) < \delta \;\Rightarrow\; \left\{\int_A \varphi_\lambda < \varepsilon,\ \forall\lambda\right\};$$

(ii) $\forall\varepsilon > 0, \exists\,G \in \mathcal{B}_+$ *such that*

$$\mu(G) < \infty \quad \textit{and} \quad \left\{\int_{\mathbb{R}_+\setminus G} \varphi_\lambda < \varepsilon,\ \forall\lambda\right\}.$$

Proof. This is a version of the famous Dunford–Pettis Theorem (Dunford and Pettis, 1940). □

The next lemma establishes part (i) of the Dunford–Pettis condition for an arbitrary trajectory starting in $\mathcal{D}$.

Lemma 13. *Assume the conditions of Proposition 10. Let $\mathcal{D}$ be as in Lemma 11. Given any $\varphi \in \mathcal{D}$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$A \in \mathcal{B}_+ \quad \textit{and} \quad \mu(A) < \delta \;\Rightarrow\; \left\{\int_A P^t\varphi < \varepsilon, \quad \forall t \geq 0\right\}. \tag{25}$$

Proof. Let V be the Lyapunov function of Condition 1. Define $E(V|g) = \int Vg$. By (7) and Fubini's theorem,

$$\begin{aligned} E(V|P^t\varphi) &= \int V(y)P^t\varphi(y)\,\mathrm{d}y = \int V(y)\left[\int p(x,y)P^{t-1}\varphi(x)\,\mathrm{d}x\right]\mathrm{d}y \\ &= \int \left[\int V(y)p(x,y)\,\mathrm{d}y\right] P^{t-1}\varphi(x)\,\mathrm{d}x. \end{aligned}$$

But

$$\int V(y)p(x, y)\,\mathrm{d}y = \int V[g(x)z]\psi(z)\,\mathrm{d}z \leq \alpha V(x) + C$$

for all x by hypothesis. Therefore,

$$E(V|P^t\varphi) \leq \int [\alpha V(x) + C]P^{t-1}\varphi(x)\,\mathrm{d}x = \alpha E(V|P^{t-1}\varphi) + C.$$

Repeating this argument obtains

$$E(V|P^t\varphi) \leq \alpha^t E(V|\varphi) + \frac{C}{1-\alpha} \leq E(V|\varphi) + \frac{C}{1-\alpha}, \quad \forall t \geq 0.$$

The right-hand side of this inequality is finite by (24).

On the other hand, it can be verified that for arbitrary positive a,

$$a\int_{\mathbb{R}_+\setminus G_a} P^t\varphi \leq E(V|P^t\varphi),$$

when G_a is defined as the set of $x \in \mathbb{R}_+$ with $V(x) \leq a$. Therefore,

$$\int_{\mathbb{R}_+\setminus G_a} P^t\varphi \leq \frac{1}{a}\left(E(V|\varphi) + \frac{C}{1-\alpha}\right), \quad \forall t \geq 0, \quad \forall a > 0. \tag{26}$$

Choose a so large that

$$\frac{1}{a}\left(E(V|\varphi) + \frac{C}{1-\alpha}\right) < \frac{\varepsilon}{2}. \tag{27}$$

Consider now the decomposition

$$\int_A P^t\varphi = \int_{A\cap G_a} P^t\varphi + \int_{A\cap[\mathbb{R}_+\setminus G_a]} P^t\varphi.$$

Using (26) and (27) gives

$$\int_A P^t\varphi \leq \int_{A\cap G_a} P^t\varphi + \frac{\varepsilon}{2}, \quad \forall t \geq 0. \tag{28}$$

It remains to bound the first term in the sum on the right-hand side of (28), taking the constant a as given—determined in (27)—and assuming that at least one of Conditions 3 or 4 holds. We break the argument down into three separate cases. □

Case I. Assume first that Condition 3 holds. Using the expression for the density kernel given in (6), for any $y > 0$,

$$\begin{aligned} P^t\varphi(y) &= \int p(x, y)P^{t-1}\varphi(x)\,\mathrm{d}x = \int \psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}P^{t-1}\varphi(x)\,\mathrm{d}x \\ &= \int \psi\left(\frac{y}{g(x)}\right)\frac{y}{g(x)}\frac{1}{y}P^{t-1}\varphi(x)\,\mathrm{d}x \leq \frac{M}{y}, \end{aligned}$$

where M is the constant in Condition 3. Therefore,

$$\int_{A\cap G_a} P^t\varphi(y)\,\mathrm{d}y \le \int_{A\cap G_a} \frac{M}{y}\,\mathrm{d}y \le \int_A J(a)\,\mathrm{d}y = J(a)\mu(A),$$

where $J(a) < \infty$ is the maximum of M/y over compact interval $G_a \subset \mathbb{R}_+ \setminus \{0\}$.

Now pick any positive δ satisfying $\delta \le \varepsilon/(J(a)2)$. For such a δ we have

$$\mu(A) < \delta \;\Rightarrow\; \int_{A\cap G_a} P^t\varphi < \frac{\varepsilon}{2}.$$

Combining this with (28) proves (25) when Condition 3 holds.

Case II. We now establish (25) when Condition 4 holds—again by bounding the first term in the sum (28)—supposing for the moment that there exists a c with $0 < c \le g(x)$ for *every* x in $\mathbb{R}_+$. In this case,

$$\begin{aligned}\int_{A\cap G_a} P^t\varphi(y)\,\mathrm{d}y &= \int_{A\cap G_a}\int p(x,y)P^{t-1}\varphi(x)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int\left[\int_{A\cap G_a} p(x,y)\,\mathrm{d}y\right]P^{t-1}\varphi(x)\,\mathrm{d}x\\ &= \int\left[\int_{A\cap G_a}\psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}\,\mathrm{d}y\right]P^{t-1}\varphi(x)\,\mathrm{d}x.\end{aligned}$$

A change of variable now gives

$$\int_{A\cap G_a} P^t\varphi(y)\,\mathrm{d}y = \int\left[\int_{g(x)^{-1}A\cap G_a}\psi(z)\,\mathrm{d}z\right]P^{t-1}\varphi(x)\,\mathrm{d}x. \tag{29}$$

Moreover, for integrable functions such as ψ it is well known that

$$\exists\,\delta' > 0 \quad \text{such that} \quad \mu(B) < \delta' \;\Rightarrow\; \int_B \psi(z)\,\mathrm{d}z < \frac{\varepsilon}{2}.$$

Therefore, setting $\delta \equiv \delta' c$ gives

$$\mu(A) < \delta \;\Rightarrow\; \left\{\int_{g(x)^{-1}A\cap G_a}\psi(z)\,\mathrm{d}z < \frac{\varepsilon}{2}, \quad \forall x \ge 0\right\}, \tag{30}$$

because

$$\mu\left(\frac{A\cap G_a}{g(x)}\right) = \frac{1}{g(x)}\mu(A\cap G_a) \le \frac{1}{c}\mu(A) < \delta'.$$

Together (29) and (30) yield

$$\mu(A) < \delta \;\Rightarrow\; \int_{A\cap G_a} P^t\varphi < \frac{\varepsilon}{2}.$$

Again, combining this with (28) gives (25).

Case III. Finally, suppose to the contrary that while Condition 4 is satisfied, there exists no c with $0 < c \leq g(x)$ for all $x \in \mathbb{R}_+$. From Condition 4 it follows then that $g(x) \downarrow 0$ as $x \downarrow 0$, and hence there exists a $d > 0$ with the property

$$\int_{g(x)^{-1}A \cap G_a} \psi(z)\,\mathrm{d}z < \frac{\varepsilon}{2} \quad \text{for almost all} \quad x \in [0, d), \tag{31}$$

owing to the fact that $\inf A \cap G_a > 0$.

Regarding $x \geq d$, evidently $g(x) \geq c' \equiv \min[g(d), b] > 0$, where b is the positive constant in Condition 4.[15] In addition, by an argument similar to that given above for Case II,

$$\mu(A) < \delta \;\Rightarrow\; \left\{ \int_{g(x)^{-1}A \cap G_a} \psi(z)\,\mathrm{d}z < \frac{\varepsilon}{2}, \quad \forall x \geq d \right\}, \tag{32}$$

where in this case $\delta \equiv \delta' c'$. Combining (29), (31) and (32) yields

$$\mu(A) < \delta \;\Rightarrow\; \int_{A \cap G_a} P^t \varphi < \frac{\varepsilon}{2}.$$

Once again, combining this with (28) implies (25).

The final lemma in the proof of Proposition 10 establishes part (ii) of the Dunford–Pettis condition.

Lemma 14. *Let φ be as in Lemma 13. For all $\varepsilon > 0$, there exists a $G \in \mathcal{B}_+$ such that $\mu(G) < \infty$ and*

$$\int_{\mathbb{R}_+ \setminus G} P^t \varphi < \varepsilon, \quad \forall t \geq 0.$$

Proof. We have already shown that

$$\int_{\mathbb{R}_+ \setminus G_a} P^t \varphi \leq \frac{1}{a} \left(E(V|\varphi) + \frac{C}{1-\alpha} \right)$$

for all positive a, all $t \geq 0$. But this inequality is sufficient, because $E(V|\varphi)$ is finite and G_a is always bounded. □

Acknowledgements

The author thanks Peter Bardsley, John Creedy, Rabee Tourky, Cuong Le Van and seminar participants at Centre de Recherche en Mathématiques, Statistique et Economie Mathématique, Université Paris 1, Panthéon-Sorbonne for helpful comments. The author would also like to thank the Economic Theory Center, University of Melbourne and the Melbourne Research Grants Scheme for financial support, as well as both Tilburg University and the

[15] Here $g(d) > 0$ by Condition 4 and the almost everywhere positivity of g.

Institute of Mathematics, Silesian University, Katowice for their hospitality during the time that this research was completed.

Appendix A

Proof of Lemma 9. Pick any two densities $\varphi \neq \varphi'$. Evidently, the function $\varphi - \varphi'$ is both strictly positive on a set of positive measure and strictly negative on a set of positive measure. Pick any $y \in \mathbb{R}_+$. By Condition 2 and the representation (6), $p(x, y) > 0$ for almost all x. It follows that $x \mapsto p(x, y)[\varphi(x) - \varphi'(x)]$ is also strictly positive on a set of positive measure and strictly negative on a set of positive measure. Therefore, by the strict triangle inequality,

$$\begin{aligned}\|P\varphi - P\varphi'\| &= \|P(\varphi - \varphi')\| = \int \left| \int p(x, y)[\varphi(x) - \varphi'(x)]\,\mathrm{d}x \right| \mathrm{d}y \\ &< \iint |p(x, y)[\varphi(x) - \varphi'(x)]|\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint p(x, y)|\varphi(x) - \varphi'(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint p(x, y)\,\mathrm{d}y|\varphi(x) - \varphi'(x)|\,\mathrm{d}x = \|\varphi - \varphi'\|,\end{aligned}$$

as was to be proved. □

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