MONOTONE MIXING AND DISTRIBUTION DYNAMICS

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Many economic applications establish stability of Markov dynamics using the monotone mixing condition (MMC) of Hopenhayn and Prescott (1992). Working in the same setting as that paper, we introduce a weak monotone mixing condition (WMMC), phrased in terms of the reversal of rank between populations, and show that it is strictly weaker than the MMC and both necessary and sufficient for global stability under the Kolmogorov metric. We apply these results to a small open economy version of the Aiyagari model, showing how the WMMC can be used to establish global stability in a setting where the MMC fails. In addition, we show that, when the state space is one-dimensional, the MMC and WMMC coincide, implying that the original MMC is necessary as well as sufficient in this setting.

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1. INTRODUCTION

Hopenhayn and Prescott (1992) showed that, when Markov transition dynamics are monotone, global stability follows from a simple mixing condition. This monotone mixing condition (MMC) has become a standard approach to establishing stability properties over a large range of applications, including international trade, human capital, business cycles, inequality, intergenerational mobility, and labor markets.[1]

In this note we extend Hopenhayn and Prescott (1992). Working in the same environment, we introduce a weak monotone mixing condition (WMMC), stated in terms of partial rank reversal between populations evolving from different initial conditions. We then show that this condition is both necessary and sufficient for global stability, and that it delivers an explicit geometric rate of convergence. In addition, we show that the WMMC is strictly weaker than the MMC, and that this matters for economic applications with multiple state variables, such as a small open economy version of the model of Aiyagari (1994). As a further contribution, we show that the WMMC and the MMC agree when the state is one-dimensional. This implies that the original MMC is in fact necessary as well as sufficient in such settings.

In terms of techniques, one factor driving our results is that we use the Kolmogorov metric to define global stability, rather than the notion of weak convergence applied by Hopenhayn and Prescott (1992). The key benefit of the Kolmogorov metric is that it respects the underlying

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[1] See, e.g., Marcet et al. (2007), Antunes and Cavalcanti (2007), Morand and Reffett (2007), Hidalgo-Cabrillana (2009), Samaniego (2008), Le Grand and Ragot (2022), Light and Weintraub (2022), Balbus et al. (2025), and Kam et al. (2025). In the context of wealth and income dynamics, Ríos-Rull (1998) calls the MMC "the American Dream and the American Nightmare" condition, since it requires that the poorest agents have positive probability of becoming rich and the richest have positive probability of becoming poor.

order on the state space. Convergence in the Kolmogorov metric is stronger than weak convergence in the setting of Hopenhayn and Prescott (1992), which assists our necessity results and makes our sufficiency results stronger. The Kolmogorov metric is often used by econometricians, since it underpins Kolmogorov–Smirnov tests.

Hopenhayn and Prescott (1992) built on earlier work by Razin and Yahav (1979), Bhattacharya and Lee (1988), and Stokey et al. (1989). Numerous authors have since considered global stability in monotone environments similar to those studied by Hopenhayn and Prescott (1992), often with the goal of weakening assumptions on the state process or the state space. (See, for example, Kamihigashi and Stachurski (2014), Foss et al. (2018), Kamihigashi and Stachurski (2019), Light and Weintraub (2022), Light (2026), or Foss and Scheutzow (2026).) To the best of our knowledge, this paper is the first to obtain an exact characterization of stability (necessity and sufficiency) in terms of monotone mixing.

2. SETUP

Our setting is that of Hopenhayn and Prescott (1992). In particular, the state space X is a compact metric space with Borel sets $\mathcal{B}$, endowed with a closed partial order $\preceq$, and X has both a least element a and a greatest element b. Given $c, d \in \mathsf{X}$ we write $[c, d]$ for the set of $x \in \mathsf{X}$ with $c \preceq x \preceq d$. We write $b\mathsf{X}$ for the bounded Borel measurable functions from X to $\mathbb{R}$, $bc\mathsf{X}$ for the continuous functions in $b\mathsf{X}$, and $\mathcal{P}$ for the Borel probability measures on X, calling elements of $\mathcal{P}$ *distributions*. A function $h \in b\mathsf{X}$ is called *increasing* if $x \preceq x'$ implies $h(x) \leq h(x')$, and $B \in \mathcal{B}$ is called *increasing* if $\mathbb{1}_B$ is increasing. We let $i\mathcal{B}$ denote the increasing sets in $\mathcal{B}$. Also, δ_x is the probability measure concentrated at x.

Let $\mathcal{M}$ denote the finite Borel measures on X. Given $\mu, \nu \in \mathcal{M}$, we write $\mu \preceq \nu$, and say that μ is *stochastically dominated* by ν, if $\mu(\mathsf{X}) = \nu(\mathsf{X})$ and $\mu(I) \leq \nu(I)$ for all $I \in i\mathcal{B}$. On $\mathcal{P}$ this is the standard notion, equivalent to $\int h \, \mathrm{d}\mu \leq \int h \, \mathrm{d}\nu$ for all increasing $h \in b\mathsf{X}$. Note that $x \preceq x'$ implies $\delta_x \preceq \delta_{x'}$. We also use the pointwise order on $\mathcal{M}$, writing $\mu' \leq \mu$ when $\mu'(B) \leq \mu(B)$ for all $B \in \mathcal{B}$ and calling μ' a *subpopulation* of μ.

A *stochastic kernel* on $(\mathsf{X}, \mathcal{B})$ is a map $P \colon \mathsf{X} \times \mathcal{B} \to [0, 1]$ such that $x \mapsto P(x, B)$ is Borel measurable for each $B \in \mathcal{B}$, while $B \mapsto P(x, B)$ is a probability measure for each $x \in \mathsf{X}$. We write $P_x := P(x, \cdot)$ and set

$$(\mu P)(B) := \int P(x, B) \, \mu(\mathrm{d}x) \qquad (B \in \mathcal{B}),$$

so that $\mu \mapsto \mu P$ maps $\mathcal{P}$ into itself. We write P^t for the t-th iterate of P, with $P^0_x := \delta_x$. A kernel P is called *increasing* if $\mu \preceq \nu$ implies $\mu P \preceq \nu P$ or, equivalently, if $P_x \preceq P_{x'}$ whenever $x \preceq x'$ (Kamae et al., 1977, Proposition 1). Evidently P^t is increasing for all t whenever P is.

A distribution $\mu^* \in \mathcal{P}$ is called *stationary* for P if $\mu^* = \mu^* P$. We call P *globally stable* if P has a unique stationary distribution $\mu^* \in \mathcal{P}$ and, in addition, $\kappa(\mu P^t, \mu^*) \to 0$ as $t \to \infty$ for every $\mu \in \mathcal{P}$, where κ is the *Kolmogorov metric*

$$\kappa(\mu, \nu) := 2 \sup_{I \in i\mathcal{B}} |\mu(I) - \nu(I)| \qquad (\mu, \nu \in \mathcal{P}). \tag{1}$$

Since X is compact, convergence in κ implies the weak convergence used in Hopenhayn and Prescott (1992); see, e.g., Kamihigashi and Stachurski (2019).

3. MAIN RESULT

Throughout this section and the next, P is an increasing stochastic kernel on $(\mathsf{X}, \mathcal{B})$. We introduce a weak monotone mixing condition and show that it characterizes global stability. Major proofs are deferred to Section 6.

Fix μ and ν in $\mathcal{P}$. It is helpful to regard μ and ν as two populations of equal size, each distributed over the state space X. A point of X is a possibly multidimensional list of attributes of an individual and $\preceq$ ranks attribute bundles. We say that μ is *partially dominated* by ν if there exist $\varepsilon > 0$ and μ', ν' in $\mathcal{M}$ with

$$\mu' \leq \mu, \qquad \nu' \leq \nu, \qquad \mu'(\mathsf{X}) = \nu'(\mathsf{X}) = \varepsilon \qquad \text{and} \qquad \mu' \preceq \nu'. \tag{2}$$

In words, a fraction ε of the population μ can be matched with an equally large fraction of ν in such a way that the first ranks weakly below the second. When (2) holds for a given ε, we say that μ is partially dominated by ν *at level* ε.[2]

Partial dominance can often be verified with a single threshold, as the next lemma shows. Its proof is given in Section 6.

LEMMA 1: *Let $\mu, \nu \in \mathcal{P}$.*

(i) *If there is an $s \in \mathsf{X}$ with $\mu([a,s]) > 0$ and $\nu([s,b]) > 0$, then μ is partially dominated by ν at level* $\min\{\mu([a,s]),\, \nu([s,b])\}$.

(ii) *If $\mathsf{X} = [a,b] \subset \mathbb{R}$ with the usual order, then the converse also holds: μ is partially dominated by ν if and only if there is an $s \in \mathsf{X}$ with $\mu([a,s]) > 0$ and $\nu([s,b]) > 0$.*

The idea behind part (i) is that the part of μ lying at or below s ranks weakly below the part of ν lying at or above s. Scaling the larger of these two subpopulations down to the size of the smaller yields μ' and ν' satisfying (2).

Figure 1 illustrates Lemma 1 in the univariate case, for two distributions with continuous CDFs F_μ and F_ν. Since $\mu([a,s]) = F_\mu(s)$ and $\nu([s,b]) = 1 - F_\nu(s)$, the threshold condition becomes $F_\mu(s) > 0$ and $F_\nu(s) < 1$. In the figure, $F_\mu \leq F_\nu$, so $\nu \preceq \mu$: in aggregate, μ ranks above ν. Nonetheless, the threshold s in the left panel satisfies this condition, so μ is partially dominated by ν at level $\varepsilon = \min\{F_\mu(s),\, 1 - F_\nu(s)\}$. The right panel shows the two subpopulations from part (i). More generally, part (ii) says that, in the univariate case, partial dominance fails only when the two populations are completely separated, in the sense that almost every member of μ ranks strictly above almost every member of ν.

Now let's turn to mixing. We say that P satisfies the *weak monotone mixing condition* (WMMC) if there exists an $n \in \mathbb{N}$ such that P^n_b is partially dominated by P^n_a. To interpret this definition, observe that, since $a \preceq b$ and P is increasing, we have $P^n_a \preceq P^n_b$ at every n. This means that the descendants of the "high class" initial condition b stochastically dominate the descendants of the "low class" initial condition a at every horizon. At the same time, this statement is made at the level of distributions and, as Figure 1 illustrates, does not rule out reversal at the level of individuals. The WMMC says that such a reversal occurs at some horizon, for a positive fraction of each population.

For the WMMC to hold, the reversal need only be weak: it suffices that the low-class descendants catch up with the high-class descendants, and no part of either population need strictly overtake the other.

We can now state our main theorem.

[2] A pair (μ', ν') satisfying (2) is called an ordered component pair in Kamihigashi and Stachurski (2019). The partial dominance terminology comes from Kamihigashi and Stachurski (2020).

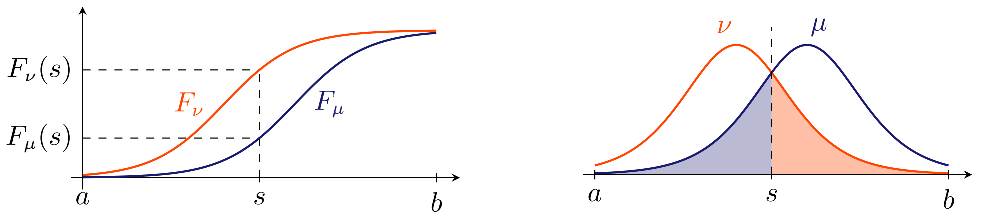

FIGURE 1.—Partial dominance of μ by ν when $\mathsf{X} \subset \mathbb{R}$. Left: CDFs, with $F_\mu(s) > 0$ and $F_\nu(s) < 1$. Right: densities, with the fraction $\mu([a,s]) = F_\mu(s)$ of μ lying at or below s and the fraction $\nu([s,b]) = 1 - F_\nu(s)$ of ν lying at or above s shaded.

THEOREM 2: *Let P be an increasing stochastic kernel on $(\mathsf{X}, \mathcal{B})$. Then P is globally stable if and only if P satisfies the weak monotone mixing condition. In this case, if $n \in \mathbb{N}$ and $\varepsilon > 0$ are such that P_b^n is partially dominated by P_a^n at level ε, then the stationary distribution μ^* obeys*

$$\kappa(\mu P^t, \mu^*) \leq 4(1-\varepsilon)^{\lfloor t/n \rfloor} \qquad \textit{for every } \mu \in \mathcal{P} \textit{ and every } t \geq 0. \tag{3}$$

In (3), ε is a fraction of the two groups whose ranking can be reversed over n periods, and it is this fraction that governs how quickly the distribution converges.

4. RELATIONSHIP TO THE MMC

In this section we relate the WMMC to the monotone mixing condition of Hopenhayn and Prescott (1992), and show that the two coincide when the state space is one-dimensional.

The *monotone mixing condition* (MMC) of Hopenhayn and Prescott (1992) requires that there exist an $n \in \mathbb{N}$ and an $s \in \mathsf{X}$ with

$$P^n(a, [s,b]) > 0 \quad \text{and} \quad P^n(b, [a,s]) > 0. \tag{4}$$

This is the threshold condition of Lemma 1, applied with $\mu = P_b^n$ and $\nu = P_a^n$: at some common horizon, a positive fraction of the descendants of b sits at or below s, while a positive fraction of the descendants of a sits at or above it. Hence, by part (i) of Lemma 1, P_b^n is partially dominated by P_a^n at level $\min\{P^n(b,[a,s]),\, P^n(a,[s,b])\}$. Thus the MMC implies the WMMC, and Theorem 2 recovers the global stability result of Hopenhayn and Prescott (1992), together with a rate of convergence.

The two conditions differ in how the reversal of rank is certified. The MMC requires a common threshold s separating the reversed fractions of the two populations. The WMMC retains the reversal but drops the threshold.

In general, the WMMC does not imply the MMC, and this matters for economic applications with multiple state variables, as shown in Section 5. In the univariate case, however, part (ii) of Lemma 1 shows that every instance of partial dominance can be certified by a threshold. Hence the WMMC reduces to the existence of an n and an s with $P^n(b,[a,s]) > 0$ and $P^n(a,[s,b]) > 0$, which is exactly the MMC. (In terms of Figure 1, take $\mu = P_b^n$ and $\nu = P_a^n$.) As a result, Theorem 2 converts the MMC into an exact characterization of global stability.

THEOREM 3: *Let $\mathsf{X} = [a,b] \subset \mathbb{R}$ with the usual order and let P be an increasing stochastic kernel on $(\mathsf{X}, \mathcal{B})$. Then P is globally stable if and only if P satisfies the* MMC.

Equivalently, by the discussion following Lemma 1, such a P fails to be globally stable if and only if, at every horizon n, almost every descendant of b ranks strictly above almost every descendant of a.

5. APPLICATION: A SMALL OPEN ECONOMY MODEL

We now apply Theorem 2 to a small open economy version of the Aiyagari (1994) model. The distinguishing feature is that the interest rate and the wage are tied together by a decreasing factor-price relation, so the two factor prices move in opposite directions. This breaks the MMC of Hopenhayn and Prescott (1992), while the WMMC still holds.

5.1. *Household Sector and Factor Prices*

Time is discrete and there is a unit continuum of households. The economy is small open, so it takes the world interest rate as given. The net world interest rate r_t is IID with a continuous distribution supported on $[\underline{r}, \bar{r}]$. A competitive firm produces a single good with a constant-returns-to-scale technology $F(K, L)$, where K and L are the demands for capital and labor, respectively. Assume that F is twice continuously differentiable, nonnegative, and strictly increasing in each input. Capital depreciates at rate $d \in (0,1)$, and $-d < \underline{r} < \bar{r} < \infty$. Let $f(k) = F(k,1)$ denote the intensive production function, with $k = K/L$. Then $f' > 0$, and we assume in addition that $f'' < 0$ and that f' satisfies the Inada conditions $f'(0) = \infty$ and $f'(\infty) = 0$. Profit maximization gives

$$r_t + d = f'(k_t), \qquad w_t = f(k_t) - k_t f'(k_t).$$

Because $f'' < 0$, and since $\underline{r} + d > 0$, the first equation has a unique solution $k_t = \psi(r_t) \in (0, \infty)$. Substituting into the second equation yields the factor-price relation

$$w_t = W(r_t), \qquad W(r) = f(\psi(r)) - \psi(r)\big(r + d\big).$$

Moreover, $W'(r) = -\psi(r) < 0$, so the wage is a strictly decreasing function of the interest rate. We set $\underline{w} = W(\bar{r})$ and $\bar{w} = W(\underline{r})$. Note that $\underline{w} > 0$, because $f(0) \geq 0$ and $f'' < 0$ imply $f(k) - kf'(k) > 0$ for all $k > 0$. Thus (r_t, w_t) always lies on a decreasing curve

$$D := \{(r, W(r)) : r \in [\underline{r}, \bar{r}]\}.$$

Each household has a period utility function $u : \mathbb{R}_+ \to \mathbb{R}$ that is twice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$. The household aims to solve

$$\begin{aligned} &\text{maximize} && \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\text{subject to} && c_t + k_{t+1} = w_t z_t + (1 + r_t) k_t, \qquad c_t, k_{t+1} \geq 0. \end{aligned}$$

Here $\beta \in (0,1)$ is the discount factor, k_t is individual wealth held at the beginning of period t, and z_t is idiosyncratic productivity. The productivity shock has a continuous distribution supported on $[\underline{z}, \bar{z}]$, where $0 < \underline{z} < \bar{z} < \infty$, and the pair (z_t, r_t) is IID over time.

5.2. *State Space and Optimal Policy*

The state vector of a household is $x = (k, z, r, w)$. We impose the impatience condition $\beta(1 + \bar{r}) < 1$ and assume that asset holdings are restricted to $[0, \bar{k}]$, where the lower bound is the borrowing constraint and $\bar{k}$ is a finite cap on savings, which we take to be large enough that the cap never binds.[3] The state space is then

$$\mathsf{X} := [0, \bar{k}] \times [\underline{z}, \bar{z}] \times [\underline{r}, \bar{r}] \times [\underline{w}, \bar{w}], \tag{5}$$

endowed with the pointwise partial order, which is closed. Under this order, X has least element $a = (0, \underline{z}, \underline{r}, \underline{w})$ and greatest element $b = (\bar{k}, \bar{z}, \bar{r}, \bar{w})$. Note that the price pair (r_t, w_t) takes values in the curve D rather than the full rectangle $[\underline{r}, \bar{r}] \times [\underline{w}, \bar{w}]$, but the rectangle is the right state space: since W is strictly decreasing, distinct points of D are unordered, so D has no least or greatest element. Nor can w simply be dropped from the state: with state (k, z, r) and $w = W(r)$, the savings policy $g(k, z, r, W(r))$ need not be monotone in r, since a higher interest rate raises interest income but lowers the wage. The redundant coordinate w is what makes the dynamics increasing, and, as we show below, the failure of the MMC is the price. The same issue arises whenever two exogenous state variables are tied together by a decreasing relation, so the phenomenon is not special to this model.

The Bellman equation is

$$V(k, z, r, w) = \max_{0 \le k' \le \bar{k}} \left\{ u\big(wz + (1 + r)k - k'\big) + \beta \mathbb{E} V\big(k', z', r', W(r')\big) \right\}, \tag{6}$$

where the expectation is over the IID shocks (z', r'), and consumption is also required to be nonnegative. The Bellman operator is a contraction on $bc\mathsf{X}$, so (6) has a unique solution $V \in bc\mathsf{X}$. Since u is concave and the constraint set is convex, V is concave in k, and strict concavity of u then implies that the maximizer is unique. The resulting savings policy $k' = g(k, z, r, w)$ takes values in $[0, \bar{k}]$, so X is a compact invariant state space. By our choice of $\bar{k}$, we have $g(x) < \bar{k}$ for all $x \in \mathsf{X}$.

5.3. *Monotonicity, Mixing, and Global Stability*

Let P be the stochastic kernel induced by the optimal policy. Given $x = (k, z, r, w)$, the next-period state is

$$x' = \big(g(k, z, r, w),\, z',\, r',\, W(r')\big),$$

where (z', r') is drawn from the exogenous continuous law. Since $u'' < 0$, the objective in (6) has increasing differences in k' and cash-on-hand $wz + (1 + r)k$, while the constraint set is increasing in the state. Hence, by Topkis's theorem, the policy g is increasing in all four arguments. Since the next-period shocks have the same law from every state, P is an increasing stochastic kernel.

We now verify that the MMC fails. Fix $n \ge 1$ and a candidate threshold $s = (s_k, s_z, s_r, s_w) \in \mathsf{X}$, and consider the sets

$$A := D \cap \{(r, w) : (r, w) \le (s_r, s_w)\} \quad \text{and} \quad B := D \cap \{(r, w) : (r, w) \ge (s_r, s_w)\}.$$

[3] The cap serves only to compactify the state space. Under impatience and standard additional restrictions on preferences, optimal wealth accumulation is bounded, so $\bar{k}$ can be chosen sufficiently large that optimal wealth always lies in the interior of $[0, \bar{k}]$; see, e.g., Açıkgöz (2018, Proposition 4) and Zhu (2020, Propositions 7–9).

If $(r,w) \in A$, then $r \leq s_r$ and $W(r) \leq s_w$; since W is decreasing, the first inequality gives $W(r) \geq W(s_r)$, so A is nonempty only if $s_w \geq W(s_r)$. A symmetric argument shows that B is nonempty only if $s_w \leq W(s_r)$. Hence, if both sets are nonempty, then $s_w = W(s_r)$ and, because W is strictly decreasing, $A = B = \{(s_r, W(s_r))\}$. At the same time, the (r,w)-marginal of P^n_x is the law of $(r, W(r))$, which is supported on D and atomless. Since $P^n(b,[a,s]) > 0$ requires $(r_n, w_n) \in A$ with positive probability, and $P^n(a,[s,b]) > 0$ requires $(r_n, w_n) \in B$ with positive probability, these two conditions cannot both hold. Thus the MMC fails at every n and every s.

We now show that the WMMC holds. The first step is to show that the borrowing constraint binds with positive probability.

LEMMA 4: *There exist $n \in \mathbb{N}$ and $p > 0$ such that $\mathbb{P}\{k_n = 0 \,|\, x_0 = b\} \geq p$.*

PROOF: Let $c(x) := wz + (1+r)k - g(x)$ be the consumption policy. It is continuous by the theorem of the maximum. Moreover, $wz \geq \underline{w}\,\underline{z} > 0$ at every $x \in \mathsf{X}$, so the Inada condition implies $c(x) > 0$; since X is compact, $\underline{c} := \min_{x \in \mathsf{X}} c(x) > 0$. Suppose, contrary to the claim, that $\mathbb{P}\{k_n = 0 \,|\, x_0 = b\} = 0$ for every $n \in \mathbb{N}$. Then, starting from b, we have $0 < k_{t+1} < \bar{k}$ almost surely at every date, so the Euler equation holds with equality: $u'(c_t) = \beta\, \mathbb{E}_t[(1 + r_{t+1})\, u'(c_{t+1})]$. Iterating gives

$$u'(c_0) = \mathbb{E}_0 \left[\prod_{i=1}^{n} \beta(1+r_i)\, u'(c_n) \right] \leq (\beta(1+\bar{r}))^n\, u'(\underline{c}) \qquad (n \in \mathbb{N}).$$

Since $\beta(1+\bar{r}) < 1$, letting $n \to \infty$ gives $u'(c_0) \leq 0$, a contradiction. *Q.E.D.*

In words, Lemma 4 states that a household that begins with maximal wealth and productivity exhausts its assets within n periods with positive probability. This is a standard scenario in Aiyagari-style models with impatient households: a sufficiently long run of unfavorable shocks drives wealth down to the borrowing constraint. See, e.g., Açıkgöz (2018, Lemma F.2), Zhu (2020, Proposition 11), and Ma et al. (2020, Lemma C.2).

Since the shocks (z_t, r_t) are IID, the time-n state consists of k_n, which depends only on x_0 and the shocks up to date $n-1$, and (z_n, r_n, w_n), which is independent of both. Hence

$$P^n_x = \lambda^x_n \otimes \eta \qquad (x \in \mathsf{X}), \tag{7}$$

where $\otimes$ denotes the product of measures, λ^x_n is the distribution of k_n given $x_0 = x$, and η is the common law of $(z, r, W(r))$. Recall from Section 3 that $P^n_a \preceq P^n_b$. The set $\{x \in \mathsf{X} : k > 0\}$ is increasing, so it receives no more mass under P^n_a than under P^n_b. Taking complements, $\lambda^a_n(\{0\}) \geq \lambda^b_n(\{0\}) \geq p$, where the second inequality holds by Lemma 4. Now let $\mu' = \nu' := p\,(\delta_0 \otimes \eta)$, where δ_0 is the unit mass at $k = 0$. By (7) and the previous display, $\mu' \leq P^n_b$ and $\nu' \leq P^n_a$, while $\mu' \preceq \nu'$ holds trivially because the two measures are identical. Hence P^n_b is partially dominated by P^n_a at level $p > 0$. Thus P satisfies the WMMC and, by Theorem 2, is globally stable. Moreover, the rate bound (3) holds with this n and $\varepsilon = p$.

In summary, the WMMC holds because its reversal need only be weak, so coalescence at the borrowing constraint suffices, while the MMC fails because prices lie on the decreasing curve D, so no single threshold s exists.

6. PROOFS

This section collects the proofs. We begin with part (i) of Lemma 1.

PROOF OF LEMMA 1, PART (I): Suppose that $s \in \mathsf{X}$ satisfies $\mu([a,s]) > 0$ and $\nu([s,b]) > 0$. Let $p := \mu([a,s])$, $q := \nu([s,b])$, and $\varepsilon := \min\{p, q\}$. Define μ' and ν' by

$$\mu'(B) := \frac{\varepsilon}{p}\,\mu(B \cap [a,s]) \quad \text{and} \quad \nu'(B) := \frac{\varepsilon}{q}\,\nu(B \cap [s,b]) \qquad (B \in \mathcal{B}). \tag{8}$$

One easily confirms that $\mu' \leq \mu$, $\nu' \leq \nu$ and $\mu'(\mathsf{X}) = \nu'(\mathsf{X}) = \varepsilon$. To see that $\mu' \preceq \nu'$, fix $I \in i\mathcal{B}$. If $s \in I$, then $[s,b] \subseteq I$ because I is increasing, so $\nu'(I) = \varepsilon$, while $\mu'(I) \leq \varepsilon$. If $s \notin I$, then $I \cap [a,s] = \varnothing$, since $y \in I$ and $y \preceq s$ would force $s \in I$, so $\mu'(I) = 0$. In both cases $\mu'(I) \leq \nu'(I)$. The claim follows. *Q.E.D.*

The key tool in the proofs of Theorems 2 and 3 is the ordered affinity of Kamihigashi and Stachurski (2019), which measures the extent to which one distribution is partially dominated by another. The *ordered affinity* of $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$ is the largest level at which partial dominance holds:

$$\alpha(\mu,\nu) := \max\{\varepsilon \in [0,1] : \mu \text{ is partially dominated by } \nu \text{ at level } \varepsilon\}. \tag{9}$$

The maximum is attained (Kamihigashi and Stachurski, 2019, Proposition 3.1). (At $\varepsilon = 0$, (2) holds trivially with $\mu' = \nu' = 0$, so the set in (9) is nonempty.) In this language, the WMMC of Section 3 states that $\alpha(P^n_b, P^n_a) > 0$ for at least one $n \in \mathbb{N}$, and, by attainment, the sharpest version of the rate bound (3) at horizon n sets $\varepsilon = \alpha(P^n_b, P^n_a)$.

We make frequent use of the dual expression

$$\alpha(\mu,\nu) = 1 - \sup_{I \in i\mathcal{B}} (\mu(I) - \nu(I)) \qquad (\mu, \nu \in \mathcal{P}), \tag{10}$$

which is part of Theorem 3.1 of Kamihigashi and Stachurski (2019). One immediate consequence is the monotonicity property

$$\hat{\mu} \preceq \mu \text{ and } \nu \preceq \hat{\nu} \quad \implies \quad \alpha(\mu,\nu) \leq \alpha(\hat{\mu},\hat{\nu}), \tag{11}$$

valid for all $\mu, \nu, \hat{\mu}, \hat{\nu} \in \mathcal{P}$. Indeed, the hypotheses of (11) give $\hat{\mu}(I) - \hat{\nu}(I) \leq \mu(I) - \nu(I)$ for every $I \in i\mathcal{B}$, and taking suprema in (10) yields the conclusion.

Finally, we make use of the *total ordered variation* distance between μ and ν, which is

$$\gamma(\mu,\nu) := 2 - \alpha(\mu,\nu) - \alpha(\nu,\mu). \tag{12}$$

That γ is a metric on $\mathcal{P}$ is Lemma 4.1 of Kamihigashi and Stachurski (2019). Moreover, γ is equivalent to κ. Indeed, (10) gives $\gamma(\mu,\nu) = \sup_{I \in i\mathcal{B}}(\mu(I) - \nu(I)) + \sup_{I \in i\mathcal{B}}(\nu(I) - \mu(I))$, where both suprema are nonnegative because $\varnothing \in i\mathcal{B}$, while (1) says that $\kappa(\mu,\nu)$ is twice the maximum of the same two suprema. Hence

$$\gamma(\mu,\nu) \leq \kappa(\mu,\nu) \leq 2\gamma(\mu,\nu) \qquad (\mu, \nu \in \mathcal{P}). \tag{13}$$

We can now prove the main theorem.

PROOF OF THEOREM 2: Suppose first that P is globally stable, and let μ^* be its stationary distribution. Since $P^t_b = \delta_b P^t$ and $P^t_a = \delta_a P^t$, global stability gives

$$\kappa(P^t_b, P^t_a) \leq \kappa(P^t_b, \mu^*) + \kappa(\mu^*, P^t_a) \to 0 \quad (t \to \infty).$$

Since $\alpha \le 1$, definition (12) gives $1-\alpha(P_b^t, P_a^t) \le \gamma(P_b^t, P_a^t)$, and $\gamma \le \kappa$ by (13). Hence

$$\alpha(P_b^t, P_a^t) \to 1 \quad (t \to \infty), \tag{14}$$

and in particular P satisfies the WMMC.

Suppose now that P satisfies the WMMC, and fix $n \in \mathbb{N}$ and $\varepsilon > 0$ such that P_b^n is partially dominated by P_a^n at level ε. By (9), $\alpha_n := \alpha(P_b^n, P_a^n) \ge \varepsilon > 0$. We claim that

$$\alpha(P_x^n, P_{x'}^n) \ge \alpha_n \qquad (x, x' \in \mathsf{X}). \tag{15}$$

To see this, fix $x, x' \in \mathsf{X}$. Since $x \preceq b$ and $a \preceq x'$, we have $\delta_x \preceq \delta_b$ and $\delta_a \preceq \delta_{x'}$. As P^n is increasing, it follows that $\delta_x P^n \preceq \delta_b P^n$ and $\delta_a P^n \preceq \delta_{x'} P^n$, so (11) gives

$$\alpha_n = \alpha(\delta_b P^n, \delta_a P^n) \le \alpha(\delta_x P^n, \delta_{x'} P^n) = \alpha(P_x^n, P_{x'}^n),$$

which is (15). Since equality holds at $(x, x') = (b, a)$, the infimum of $\alpha(P_x^n, P_{x'}^n)$ over $\mathsf{X} \times \mathsf{X}$ is exactly $\alpha_n > 0$. This is the hypothesis of Corollary 5.1 of Kamihigashi and Stachurski (2019), which therefore applies: the kernel P has a unique stationary distribution $\mu^* \in \mathcal{P}$, and

$$\gamma(\mu P^t, \mu^*) \le (1-\alpha_n)^{\lfloor t/n \rfloor} \gamma(\mu, \mu^*) \qquad (\mu \in \mathcal{P},\ t \ge 0). \tag{16}$$

Combining (16) with $\alpha_n \ge \varepsilon$ and (13) gives

$$\kappa(\mu P^t, \mu^*) \le 2\gamma(\mu P^t, \mu^*) \le 2(1-\varepsilon)^{\lfloor t/n \rfloor} \gamma(\mu, \mu^*) \le 4(1-\varepsilon)^{\lfloor t/n \rfloor}$$

for all $\mu \in \mathcal{P}$ and $t \ge 0$, where the last step uses $\gamma(\mu, \mu^*) \le \kappa(\mu, \mu^*) \le 2$. This is (3), and in particular $\kappa(\mu P^t, \mu^*) \to 0$ for every $\mu \in \mathcal{P}$, so P is globally stable. *Q.E.D.*

Now we turn to the univariate case, beginning with part (ii) of Lemma 1.

PROOF OF LEMMA 1, PART (II): Sufficiency is immediate from part (i). For necessity we argue by contraposition. Assume that, for all $s \in [a, b]$, either $\mu([a, s]) = 0$ or $\nu([s, b]) = 0$. Set

$$\tau := \sup\left(\{a\} \cup \{r \in [a, b] : \mu([a, r]) = 0\}\right).$$

Since $r \mapsto \mu([a, r])$ is nondecreasing, the set in question is an interval containing its smaller points, so $\mu([a, r]) = 0$ for every $r < \tau$ and $\mu([a, r]) > 0$ for every $r > \tau$. In particular $\mu([a, \tau)) = 0$. Define I as $[\tau, b]$ if $\mu(\{\tau\}) > 0$ and $(\tau, b]$ otherwise. In either case I is an increasing Borel subset of X with $\mu(I) = 1$. We claim that $\nu(I) = 0$. Given the claim, $\mu(I) - \nu(I) = 1$, so the supremum in (10) equals one and $\alpha(\mu, \nu) = 0$. By (9), μ is then not partially dominated by ν at any positive level, and the proof is complete.

Suppose first that $\mu(\{\tau\}) > 0$. Then $\mu([a, \tau]) > 0$, so $\nu([\tau, b]) = 0$ by hypothesis, and $I = [\tau, b]$. Suppose instead that $\mu(\{\tau\}) = 0$. Then $\mu([a, \tau]) = 0$, and since $\mu([a, b]) = 1$ this forces $\tau < b$. For every $s \in (\tau, b]$ we have $\mu([a, s]) > 0$ and hence $\nu([s, b]) = 0$. Letting s decrease to τ along $(\tau, b]$ gives $\nu(I) = \nu((\tau, b]) = 0$. *Q.E.D.*

The same argument justifies the description of part (ii) given after Lemma 1. Every point of the set I constructed above lies strictly above every point of $\mathsf{X} \setminus I$, so if μ is not partially dominated by ν, then $(\mu \otimes \nu)\{(x, y) : x > y\} = 1$. Conversely, if this equality holds, then $\mu([a, s])\,\nu([s, b]) \le (\mu \otimes \nu)\{(x, y) : x \le y\} = 0$ for every $s \in \mathsf{X}$, so μ is not partially dominated by ν by part (ii).

PROOF OF THEOREM 3: Immediate from Theorem 2 and Lemma 1, as explained in Section 4. *Q.E.D.*

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