Firm Entry and Exit with Unbounded Productivity Growth

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Abstract. In Hopenhayn's (1992) entry-exit model productivity is bounded, implying that the predicted firm size distribution cannot match the power law tail observable in the data. In this paper we remove the boundedness assumption and, in this more general setting, provide an exact characterization of existence of stationary equilibria, as well as a novel sufficient condition for existence based on treating production as a Lyapunov function. We also provide new representations of the rate of entry and aggregate supply. Finally, we prove that the firm size distribution has a power law tail under a very broad set of productivity growth specifications.
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1. Introduction

Observed productivity growth is driven by the decisions of firms. Existing firms innovate, while new firms disrupt the status quo by bringing fresh ideas. The specifics of this process have far-reaching implications for long run growth in per capita output, output volatility, employment, the income distribution, the wealth distribution, and the development of technology and human capital.

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One of the most important models of productivity growth developed in the last few decades is the entry-exit model of Hopenhayn (1992). This model forms a cornerstone of modern quantitative economics and researchers have extended the core ideas to study a broad range of topics, from technical change and development to aggregate volatility and business cycle fluctuations.[1]

In building his model, Hopenhayn (1992) makes one key technical assumption to streamline his analysis: firm productivity is bounded. This assumption simplifies modeling the exit decisions of firms (since value functions are bounded and Bellman operators are ordinary contraction mappings), as well as the proof of existence and uniqueness of competitive equilibrium, and the analysis of the stationary distribution. Much of the subsequent theoretical and quantitative work follows his assumption.

At the same time, under the bounded productivity assumption the model of Hopenhayn (1992) cannot match an important feature of the data: the firm size distribution is extremely heavy tailed – in fact a power law, with a Pareto tail coefficient near unity.[2] This power law property is significant for a range of aggregate outcomes. For example, idiosyncratic firm-level shocks generate substantial aggregate volatility when the firm size distribution has a power law.[3] Moreover, the right tail of the firm size distribution affects the income and wealth distributions, partly due to the high concentration of firm ownership and entrepreneurial equity.[4] The income and wealth distributions in turn affect other economic phenomena, including the composition of aggregate demand and the growth rate of aggregate productivity.[5]

[1]The literature is very large. Well known examples include Hopenhayn and Rogerson (1993), Clementi and Hopenhayn (2006), Ericson and Pakes (1995), Luttmer (2011), Acemoglu and Cao (2015), Cao et al. (2019), and Carvalho and Grassi (2019).

[2]In other words, for some measure of firm size S, there are positive constants k and α close to one such that $\mathbb{P}\{S > s\} \approx ks^{-\alpha}$ for large s. A well known reference is Axtell (2001). The power law finding has been replicated in many studies. See, for example, Gaffeo et al. (2003), who treats the G7 economies, as well as Cirillo and Hüsler (2009), Kang et al. (2011) and Zhang et al. (2009), who use Italian, Korean and Chinese data respectively.

[3]See, for example, Nirei (2006), Gabaix (2011), and Carvalho and Grassi (2019).

[4]See, for example, Benhabib and Bisin (2018). The impact of capital income on income and wealth dispersion has risen in recent years, as documented and analyzed in Kacperczyk et al. (2018).

[5]The literature on the connection between of the distribution of income and wealth and growth rates is extensive. A recent example combining theory and empirics is Halter et al. (2014).

One additional advantage of modeling the power law in the firm size distribution is calibration and testing: matching the Pareto tail index in the data serves as a valuable additional restriction to fit parameters.

At the same time, analyzing the entry-exit model of Hopenhayn (1992) without the boundedness assumption on productivity is nontrivial. One reason is that the lifetime profits of firms are potentially unbounded, requiring a new approach to the firm decision problem. Second, the time invariance condition for the equilibrium measure of firms concerns stationary distributions of Markov transitions that are possibly transient, due to the unboundedness of productivity. Third, aggregate output is potentially infinite, since integration across productivity states is over an unbounded set.

In this paper we provide a comprehensive analysis of the entry-exit model of Hopenhayn (1992) without the boundedness assumption. We show that the complications described above can be cleanly handled by using (i) a weighted supremum norm for the firm decision problem and (ii) Kac's Theorem on positive recurrence to handle productivity dynamics. Through this combination, we provide an exact necessary and sufficient condition for existence and uniqueness of a stationary recursive equilibrium in the unbounded setting. In particular, we show that a stationary recursive equilibrium exists if and only if the expected lifetime output of firms is finite. This generalizes the result in Hopenhayn (1992), where expected lifetime output is automatically finite under the stated restrictions on the productivity process.

Since expected lifetime output is endogenous, we also provide a sufficient condition based on a drift restriction over productivity dynamics. Drift conditions are a well-known technique for controlling Markov processes on unbounded state spaces (see, e.g., Meyn and Tweedie (2012)), with the main idea being to obtain a Lyapunov function on the state space such that (a) the function becomes large as the state diverges and (b) the value assigned to the state by the Lyapunov function under the Markov process in question tends to decrease if the state variable begins to diverge. The main difficulty with the approach is finding a suitable Lyapunov function. The innovation introduced in this paper is to use firm output itself as the Lyapunov function. The resulting drift condition is weak enough to allow a very large range of specifications for incumbent productivity growth.

In addition, under the stated lifetime condition, we provide a decomposition of the equilibrium firm size distribution and a sample path interpretation via Pitman's occupation measure. The latter connects the cross-sectional mass of firms in a given region of the distribution with the occupation times of individual firms. Using this decomposition, we prove a new formula connecting aggregate supply (and hence aggregate demand) with the equilibrium entry rate and the expected lifetime output of firms.

The proof of existence of a stationary recursive equilibrium is constructive, so quantitative tractability of the entry-exit model is preserved.

With these results in place, we then turn to studying Pareto tails in the firm size distribution. We analyze a setting that admits a broad range of specifications of firm-level dynamics, including those that follow Gibrat's law – a commonly used baseline – and those with the systematic departures from Gibrat's law. (For example, small firms can grow faster than large firms and their growth rates can exhibit greater volatility.) We prove that when any of these firm-level dynamics are inserted into the Hopenhayn (1992) model described above, the endogenous firm size distribution generated by entry and exit exhibits a Pareto tail.[6]

Our results show that the Pareto tail result does not depend on the shape of the entrants' distribution, beyond a simple moment condition, or the demand side of the market. In this sense, the Pareto tail becomes a highly robust prediction of the standard entry-exit model once the state space is allowed to be unbounded. We also show that the tail index, which determines the amount of mass in the right tail of the distribution and has been the source of much empirical discussion (see, e.g., Axtell (2001) or Gabaix (2016)), depends only on the law of motion for incumbents. As such, it is invariant to the productivity distribution for new entrants, the profit functions of firms, and the structure of demand.

This paper builds on previous studies that have linked random firm-level growth within an industry to Pareto tails in the cross-sectional distribution of firm size.

[6]The results described above are valid whenever the deviation between incumbents' firm-level growth dynamics and Gibrat's law is not infinitely large, in the sense of expected absolute value. Although this restriction is surprisingly weak, it tends to bind more for large firms than for small ones, since large firms have greater weight in the integral that determines expected value. This restriction is consistent with the data, since large firms tend to conform more to Gibrat's law than do small ones (see, e.g., Evans (1987a), Evans (1987b) or Becchetti and Trovato (2002)).

Early examples include Champernowne (1953) and Simon (1955), who showed that Pareto tails in stationary distributions can arise if time series follow Gibrat's law along with a reflecting lower barrier. Since then it has been well understood that Gibrat's law can generate Pareto tails for the firm size distribution in models where firm dynamics are exogenously specified. Surveys can be found in Gabaix (2008) and Gabaix (2016).

Córdoba (2008) points out that Gibrat's law is not supported by the data on firm growth and considers a generalization where volatility can depend on firm size. He then shows that Pareto tails still arise in a discrete state setting under such dynamics. Our findings strengthens his result in two ways. First, the firm size distribution is endogenously determined as the equilibrium outcome of an entry-exit model, allowing us to consider how regulations, policies and demand impact on the distribution. Second, we allow other departures from Gibrat's law supported by the data, such as dependence of the mean growth rate on firm size.

Like this paper, Carvalho and Grassi (2019) studies heavy tails in a Hopenhayn-style entry-exit model with a large but finite number of firms. The paper provides important insights on the connection between firm-level shocks and aggregate productivity. At the same time, productivity is still bounded, like Hopenhayn (1992), and Carvalho and Grassi (2019) omit conditions under which a stationary equilibrium exists. Conditions on exogenous firm productivity growth and the entrants distribution are stricter. We enhance their power law finding while showing that the key results are invariant to the productivity distribution of new entrants.

There are a several studies not previously mentioned that generate Pareto tails for the firm size distribution using a number of alternative mechanisms. A classic example is Lucas (1978), which connects heterogeneity in managerial talent to a Pareto law. More recent examples include Luttmer (2011), Acemoglu and Cao (2015) and Cao et al. (2019). While important in their own right, none of these papers provide new results on equilibria in the Hopenhayn (1992) entry-exit model, and their techniques for generating Pareto tails are more specialized. Unlike this paper, none show that the Pareto tail is a highly robust prediction of the basic entry-exit model.

On a technical level, this paper is somewhat related to the work of Benhabib et al. (2015), who study a nonlinear process associated with optimal household savings that approximates a Kesten process when income is large. This is somewhat analogous to our treatment of the firm size distribution, in that we allow nonlinear firm-level dynamics that approximate Gibrat's law. However, the topic and underlying methodology are substantially different.

The remainder of the paper is structured as follows. Section 2 sets out the model. Section 3 shows existence of a unique stationary recursive equilibrium when the state space is unbounded. Section 4 investigates heavy tails and Section 5 concludes. Long proofs are deferred to the appendix.

2. Entry and Exit

Apart from unbounded productivity, our assumptions follow Hopenhayn (1992). There is a single good produced by a continuum of firms, consisting at each point in time of a mixture of new entrants and incumbents. The good is sold at price p and the demand is given by $D(p)$.

Assumption 2.1. The demand function D is continuous and strictly decreasing with $D(0) = \infty$ and $\lim_{p \to \infty} D(p) = 0$.

Assumption 2.1 already implies that $p = 0$ cannot be an equilibrium, since demand is infinite at that price. This assumption is convenient but can be weakened if necessary, since we show below that supply is zero when $p = 0$.

Firms facing output price p and having firm-specific productivity φ generate profits $\pi(\varphi, p)$ and produce output $q(\varphi, p)$. (We take q and π as given but provide examples below where they are derived from profit maximization problems.) Profits are negative on the boundary due to fixed costs, as in Hopenhayn (1992). In particular,

Assumption 2.2. Both π and q are continuous and strictly increasing on $\mathbb{R}^2_+$. The function q is nonnegative while π satisfies $\pi(\varphi, p) < 0$ if either $\varphi = 0$ or $p = 0$.

Productivity of each incumbent firm updates according to the idiosyncratic Markov state process $\Gamma(\varphi, \mathrm{d}\varphi')$, where Γ is a transition probability kernel on $\mathbb{R}_+$. The outside option for firms is zero and the value $v(\varphi, p)$ of of an incumbent satisfies

$$v(\varphi, p) = \pi(\varphi, p) + \beta \max \left\{ 0, \int v(\varphi', p) \Gamma(\varphi, \mathrm{d}\varphi') \right\}, \tag{1}$$

where $\beta = 1/(1 + r)$ for some fixed $r > 0$. Here and below, integrals are over $\mathbb{R}_+$.

Assumption 2.3. The productivity kernel Γ is monotone increasing. In addition,

(a) For each $a > 0$ and $\varphi \geqslant 0$, there is an $n \in \mathbb{N}$ such that $\Gamma^n(\varphi, [0, a)) > 0$.
(b) For each $p > 0$, there exists a $\varphi \geqslant 0$ such that $\int \pi(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi') \geqslant 0$.

The symbol Γ^n denotes the n-step transition kernel. The monotonicity assumption means that $\Gamma(\varphi, [0, a])$ is decreasing in φ for all $a \geqslant 0$. Condition (a) is analogous to the recurrence condition in Hopenhayn (1992). Condition (b) ensures that not all incumbents exit every period.

New entrants draw productivity independently from a fixed probability distribution γ and enter the market if $\int v(\varphi', p)\,\gamma(\mathrm{d}\varphi') \geqslant c_e$, where $c_e > 0$ is a fixed cost of entry.

Assumption 2.4. The distribution γ satisfies $\int q(\varphi, p)\gamma(\mathrm{d}\varphi) < \infty$ and puts positive mass on the interval $[0, a]$ for all $a > 0$.

The first condition in Assumption 2.4 is a regularity condition that helps to ensure finite output. The second condition is convenient because it leads to aperiodicity of the endogenous productivity process.

Assumption 2.5. There exists a $p > 0$ such that $\int \pi(\varphi, p)\,\gamma(\mathrm{d}\varphi) \geqslant c_e$.

Assumption 2.5 ensures that entry occurs when the price is sufficiently large. It is relatively trivial because, for price taking firms, revenue is proportional to price.

For realistic industry dynamics, we also need a nonzero rate of exit. We implement this by assuming that, when a firm's current productivity is sufficiently low, its expected lifetime profits are negative:

Assumption 2.6. The profit function obeys $\sum_{t\geqslant 0} \beta^t \int \pi(\varphi', p)\Gamma^t(0, \mathrm{d}\varphi') \leqslant 0$.

Assumption 2.6 clearly holds if π is bounded. Another setting where Assumption 2.6 holds is when firm growth follows Gibrat's law, so that Γ is represented by the recursion $\varphi_{t+1} = A_{t+1}\varphi_t$ for some positive IID sequence $\{A_t\}$. Then $\varphi_0 = 0$ implies $\varphi_t = 0$ for all t, and hence $\Gamma^t(0, \mathrm{d}\varphi')$ is a point mass at zero. Hence the integral in Assumption 2.6 evaluates to $\pi(0, p)$ for each t, which is negative by Assumption 2.2.

Since productivity is unbounded and profits can be arbitrarily large, we also need a condition on the primitives to ensure that v is finite. In stating it, we consider the productivity process $\{\varphi_t\}$ defined by

$$\varphi_0 \sim \gamma \text{ and } \varphi_{t+1} \sim \Gamma(\varphi_t, \mathrm{d}\varphi') \text{ when } t \geqslant 1. \tag{2}$$

Assumption 2.7. There is a $\delta \in (\beta, 1)$ with $\sum_{t \geqslant 0} \delta^t \, \mathbb{E} \, \pi(\varphi_t, p) < \infty$ at all $p \geqslant 0$.

While slightly stricter than a direct bound on lifetime profits, Assumption 2.7 has the benefit of yielding a contraction result for the Bellman operator corresponding to the Bellman equation (1). Since we are working in a setting where profits can be arbitrarily large, the value function is unbounded, so the contraction in question must be with respect to a *weighted* supremum norm. To construct this norm, we take δ as in Assumption 2.7 and let

$$\kappa(\varphi, p) := \sum_{t \geqslant 0} \delta^t \mathbb{E}_\varphi \hat{\pi}(\varphi_t, p) \quad \text{with } \hat{\pi} := \pi + b. \tag{3}$$

Here b is a constant chosen such that $\pi + b \geqslant 1$. The function κ is constructed so that it dominates the value function and satisfies $1 \leqslant \kappa < \infty$ at all points in the state space.[7] For each scalar-valued f on $\mathbb{R}^2_+$, let $\|f\|_\kappa := \sup |f/\kappa|$. This is the κ-weighted supremum norm. If it is finite for f then we say that f is κ-bounded. Let

$$\mathscr{C} := \text{all continuous, increasing and } \kappa\text{-bounded functions on } \mathbb{R}^2_+.$$

Under the distance $d(v, w) := \|w - v\|_\kappa$, the set $\mathscr{C}$ is a complete metric space.[8]

Assumption 2.8. If u is in $\mathscr{C}$, then $(\varphi, p) \mapsto \int u(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi')$ is continuous.

Assumption 2.8 is a version of the continuity property imposed by Hopenhayn (1992), modified slightly to accommodate the fact that Γ acts on unbounded functions.

3. Stationary Recursive Equilibrium

Now we turn to existence, uniqueness and computation of stationary recursive equilibria for the industry. All assumptions from the previous section are in force.

[7] To be more precise, $\varphi \mapsto \kappa(\varphi, p)$ is finite γ-almost everywhere by Assumption 2.7. If γ is supported on all of $\mathbb{R}_+$, then, since the function in question is monotone, this implies that κ is finite everywhere. If not, then we tighten the assumptions above by requiring that $\kappa(\varphi, p)$ is finite everywhere.

[8] Completeness of the set of continuous κ-bounded functions under d is proved in many places, including Hernández-Lerma and Lasserre (2012), §7.2. Our claim of completeness of $(\mathscr{C}, d)$ follows from the fact that the limit of a sequence of increasing functions in $(\mathscr{C}, d)$ is also increasing.

3.1. **Preliminary Results.** We begin our analysis with the firm decision problem. The next lemma determines the firm value function $\bar{v}$, where $\bar{v}(\varphi, p)$ is lifetime value of the firm given current productivity φ and price p.

Lemma 3.1. *The Bellman operator $T \colon \mathscr{C} \to \mathscr{C}$ defined at $v \in \mathscr{C}$ by*

$$(Tv)(\varphi, p) = \pi(\varphi, p) + \beta \max\left\{0, \int v(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi')\right\} \tag{4}$$

is a contraction of modulus β on the metric space $(\mathscr{C}, d)$. Its unique fixed point $\bar{v}$ is strictly increasing and $\bar{v}(\varphi, p) < 0$ if either $\varphi = 0$ or $p = 0$.

Given $\bar{v}$, we let $\bar{\varphi}$ be the *exit threshold* function defined by

$$\bar{\varphi}(p) := \min\left\{\varphi \geqslant 0 \;\Big|\; \int \bar{v}(\varphi', p)\, \Gamma(\varphi, \mathrm{d}\varphi') \geqslant 0\right\}. \tag{5}$$

With the convention that incumbents who are indifferent remain rather than exit, an incumbent with productivity φ exits if and only if $\varphi < \bar{\varphi}(p)$. In (5) we take the usual convention that $\min \varnothing = \infty$.

Lemma 3.2. *$\bar{\varphi}$ is finite, strictly positive and decreasing on $(0, \infty)$ with $\bar{\varphi}(0) = \infty$.*

3.2. **Definitions.** Let $\mathscr{B}$ be the Borel subsets of $\mathbb{R}_+$ and $\mathscr{M}$ be all measures on $\mathscr{B}$. Taking $\bar{v}$ and $\bar{\varphi}$ as defined in the previous section, a *stationary recursive equilibrium* is a triple

$$(p, M, \mu) \text{ in } \mathcal{E} := (0, \infty) \times (0, \infty) \times \mathscr{M},$$

with p understood as price, M as mass of entrants, and μ as a distribution of firms over productivity levels, such that the goods market clears:

$$\int q(\varphi, p)\mu(\mathrm{d}\varphi) = D(p), \tag{6}$$

the *invariance condition*

$$\mu(B) = \int \Gamma(\varphi, B)\, \mathbb{1}\{\varphi \geqslant \bar{\varphi}(p)\}\, \mu(\mathrm{d}\varphi) + M\, \gamma(B) \text{ for all } B \in \mathscr{B}, \tag{7}$$

holds, the *equilibrium entry condition*

$$\int \bar{v}(\varphi, p)\, \gamma(\mathrm{d}\varphi) = c_e \tag{8}$$

holds, and the *balanced entry and exit condition*

$$M = \mu\{\varphi < \bar{\varphi}(p)\} \tag{9}$$

is verified.

3.3. **Existence and Uniqueness.** Throughout this section, we take $\{\varphi_t\}$ as in (2) and set

$$\tau(p) := \inf\{t \geqslant 1 : \varphi_t < \bar{\varphi}(p)\}. \tag{10}$$

The random variable $\tau(p)$ records *firm lifespan* associated with productivity path $\{\varphi_t\}$ when output price is p. *Lifetime firm output* is the random variable

$$\ell(p) := \sum_{t=1}^{\tau(p)} q(\varphi_t, p). \tag{11}$$

The first result of this section provides a candidate equilibrium price, which equates the expected value of entry to its cost.

Lemma 3.3. *There exists a unique $p > 0$ such that $\int \bar{v}(\varphi, p)\,\gamma(\mathrm{d}\varphi) = c_e$.*

In what follows we let

$$p^* := \text{ the unique positive price in Lemma 3.3} \tag{12}$$

and call it the *equilibrium entry price.*

We can now state our main existence and uniqueness result, which characterizes equilibrium for the entry-exit model set out in Section 2. All assumptions from that section in force.

Theorem 3.4. *The following statements are equivalent:*

(a) $\mathbb{E}\ell(p^*) < \infty$.
(b) *There exists an $M^* \in (0, \infty)$ and μ^* in $\mathscr{M}$ such that (p^*, M^*, μ^*) is a stationary recursive equilibrium.*

If either and hence both of these statements are true, then

(i) *(p^*, M^*, μ^*) is the only stationary recursive equilibrium in $\mathcal{E}$,*
(ii) *equilibrium expected firm lifespan $\mathbb{E}\,\tau(p^*)$ is finite,*
(iii) *the equilibrium (p^*, M^*, μ^*) obeys*

$$\mu^*(B) = M^* \cdot \mathbb{E} \sum_{t=1}^{\tau(p^*)} \mathbb{1}\{\varphi_t \in B\} \quad \textit{for all } B \in \mathscr{B}, \tag{13}$$

(iv) *and aggregate supply obeys*

$$\int q(\varphi, p^*)\mu^*(\mathrm{d}\varphi) = M^* \mathbb{E}\,\ell(p^*). \tag{14}$$

The decomposition (13) ties the cross-sectional distribution of productivity to dynamics at the level of the firm. It says that the mass of firms in set B is proportional to the expected number of times that a firm's productivity visits B over its lifespan. The decomposition is obtained by a combination of Kac's Theorem and the Pitman occupation formula. One simple special case of (13) is when $B = \mathbb{R}_+$, which yields

$$\frac{M^*}{\mu^*(\mathbb{R}_+)} = \frac{1}{\mathbb{E}\,\tau(p^*)}.$$

Thus, in equilibrium, the entry rate equals the reciprocal of the expected lifespan of firms.

The result in (14) provides a simple formula connecting aggregate supply (and aggregate demand) with the equilibrium entry rate and the expected lifetime output of firms. The formula may be used for calibration or testing in quantitative work.

One special case of Theorem 3.4 is when productivity is bounded above, as in Hopenhayn (1992). To see this, suppose φ_t is bounded above by B and let $\mathbf{a}^* := [0, \bar{\varphi}(p^*))$. By Assumption 2.3 there exists an integer n such that $\varepsilon := \Gamma^n(B, \mathbf{a}^*) > 0$. Because the process $\{\varphi_t\}$ renews whenever it visits $\mathbf{a}^*$, regenerating with a fresh draw from the entry distribution γ, the process $\{\varphi_t\}$ falls below $\bar{\varphi}(p^*)$ with independent probability at least ε every n periods. As a result,

$$\mathbb{E}\,\tau(p^*) = \sum_{m \in \mathbb{N}} \mathbb{P}\{\tau(p^*) \geqslant m\} \leqslant \sum_{m \in \mathbb{N}} (1-\varepsilon)^{\lfloor m/n \rfloor} < \infty.$$

Since $q(\varphi_t, p^*) \leqslant \bar{q} := q(B, p^*)$, this implies $\mathbb{E}\,\ell(p^*) \leqslant \bar{q}\mathbb{E}\,\tau(p^*) < \infty$. In particular, bounded productivity implies (a) in Theorem 3.4, and hence (b) and (i)–(iv).

3.4. **A Sufficient Condition.** In the preceding paragraph we gave a strict sufficient condition for the conclusions of Theorem 3.4 to hold. In this section, we provide a more general sufficient condition based around the idea of using output as a Lyapunov function. The condition depends only on primitives.

Assumption 3.1. For each $p > 0$, there exists an $\lambda \in (0, 1)$ and $L < \infty$ such that

$$\int q(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi') \leqslant \lambda q(\varphi, p) + L \quad \text{for all } \varphi \geqslant 0. \tag{15}$$

Assumption 3.1 says that output growth for incumbents is expected to be negative whenever current output is sufficiently large.[9] In the literature on Markov processes, the bound in (15) is sometimes called a Foster–Lyapunov drift condition. The key idea in Assumption 3.1 is that the output function q can adopted as the Lyapunov function in the drift condition.

Let the assumptions in Section 2 hold. The following result shows that the drift condition in (15) is sufficient for finite expected firm lifetimes, and hence, by Theorem 3.4, for existence and uniqueness of a stationary recursive equilibrium.

Proposition 3.5. *If Assumption 3.1 holds, then* $\mathbb{E}\,\ell(p^*) < \infty$.

The intuition behind Proposition 3.5 is as follows. When Assumption 3.1 is in force, incumbents with sufficiently large output tend to see output fall in the next period. Output is a strictly increasing function of φ, so falling output means falling productivity. From this one can construct a finite interval such that, for any given incumbent, productivity returns to this interval infinitely often. At each such occasion, the recurrence condition in Assumption 2.3 yields an independent ε probability of exiting. Eventually the firm exits and lifetime output remains finite.[10]

Example 3.1. Suppose that incumbent productivity grows according to

$$\varphi_{t+1} = A_{t+1}\varphi_t + Y_{t+1} \quad \text{for some IID sequence } \{A_t, Y_t\}, \tag{16}$$

that production is linear in φ and that all factors of production are constant, so that $q(\varphi, p) = e\varphi$ for some $e > 0$. Regarding the drift condition (15), we have

$$\int q(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi') = e\mathbb{E}A_{t+1}\varphi + \mathbb{E}Y_{t+1} = \mathbb{E}A_{t+1}q(\varphi, p) + \mathbb{E}Y_{t+1}.$$

Assumption 3.1 is therefore satisfied whenever $\mathbb{E}A_t < 1$ and $\mathbb{E}Y_t < \infty$.

Example 3.2. Suppose instead that production is Cobb–Douglas, with output φn^θ under labor input n and parameter $\theta \in (0,1)$. With profits given by $p\varphi n^\theta - c - wn$

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[9] To see this, we can write $q(\varphi_t, p)$ as Q_t and express (15) as $\ln(\mathbb{E}_t Q_{t+1}/Q_t) \leqslant \ln(\lambda + L/Q_t)$. When Q_t is sufficiently large, the right-hand side is negative.

[10] Even if firm lifespan is finite along every sample path, this does not necessarily imply that the expectation $\mathbb{E}\tau(p^*)$ is finite. Hence there are some subtleties involved in the proof of Proposition 3.5. The reason that output is used as the Lyapunov function is that we need this expectation to be finite. The appendix gives details.

for some $c, w > 0$, the function for output at optimal labor input is

$$q(\varphi, p) = \varphi^{\eta} m(p) \quad \text{where } \eta := \frac{1}{1-\theta} \text{ and } m(p) := \left(\frac{p\theta}{w}\right)^{\theta/(1-\theta)}.$$

If productivity growth follows $\varphi_{t+1} = A_{t+1}\varphi_t$, then the right-hand side of the drift condition (15) becomes

$$\int q(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi') = \mathbb{E}(A_{t+1}\varphi)^{\eta}\, m(p) = \mathbb{E}A_{t+1}^{\eta} q(\varphi, p).$$

Thus, Assumption 3.1 is valid whenever $\mathbb{E}[A_t^{\eta}] < 1$. If, say, A_t is lognormal with $A_t = \exp(m + \sigma Z)$ for $Z \sim N(0, 1)$, then $\mathbb{E}[A_t^{\eta}] = \mathbb{E}\exp(\eta m + \eta\sigma Z) = \exp(\eta m + (\eta\sigma)^2/2)$ and the condition becomes

$$m + \frac{1}{1-\theta}\frac{\sigma^2}{2} < 0.$$

This joint restriction on the rate of incumbent firm growth and the Cobb–Douglas production parameter θ is sufficient for Assumption 3.1 and hence finite firm lifetimes.

3.5. **Computing the Solution.** For the purposes of this section, we insert balanced entry and exit into the time invariance condition, yielding

$$\mu_p(B) = \int \Pi_p(\varphi, B)\mu_p(\mathrm{d}\varphi) \quad \text{for all } B \in \mathscr{B}, \tag{17}$$

where Π_p is the transition kernel on $\mathbb{R}_+$ defined by

$$\Pi_p(\varphi, B) = \Gamma(\varphi, B)\mathbb{1}\{\varphi \geqslant \bar{\varphi}(p)\} + \mathbb{1}\{\varphi < \bar{\varphi}(p)\}\gamma(B). \tag{18}$$

In the appendix we show that there exists a unique μ_p satisfying (17) whenever firms have finite expected lifespan.

Let $\mathscr{P}$ be the Borel probability measures on $\mathbb{R}_+$. Under the finite expected lifetime condition from Theorem 3.4, the unique stationary equilibrium can be computed as follows:

- (S1) Obtain $\bar{v}$ as the unique fixed point of T in $\mathscr{C}$ and $\bar{\varphi}$ as in (5).
- (S2) Solve for the equilibrium entry price p^*, as in Lemma 3.3.
- (S3) Define Π_{p^*} via (18) and compute μ_{p^*} as the unique solution to (17) in $\mathscr{P}$.
- (S4) Rescale μ_{p^*} by setting $s := D(p^*)/\int q(\varphi, p^*)\mu_{p^*}(\mathrm{d}\varphi)$ and then $\mu^* := s\,\mu_{p^*}$.
- (S5) Obtain the mass of entrants via $M^* = \mu^*\{\varphi < \bar{\varphi}(p^*)\}$.

The proof of Theorem 3.4 in the appendix confirms that the triple (p^*, M^*, μ^*) computed via (S1)–(S5) is a stationary recursive equilibrium.

Regarding (S1), $\bar{v}$ is a fixed point of a contraction map, as shown in Lemma 3.1. This provides the basis of a globally convergent method of computation. The value p^* in (S2) can be obtained once we solve for $\bar{v}$ and $\bar{\varphi}$.

Uniqueness in (S3) always holds because Π_p in (18) is γ-irreducible (see Meyn and Tweedie (2012) for definitions) whenever $p > 0$. The condition $\mathbb{E}\tau(p^*) < \infty$ then implies that the same process is Harris recurrent and ergodic, opening avenues for computing μ_{p^*} through either simulation or successive approximations.[11]

Rescaling in (S4) is implemented so that the goods market clears.

One nontrivial issue with the computation in (S3) is that, as shown in the next section, the productivity distribution μ_{p^*} and hence the firm size distribution μ^* have very heavy tails for under realistic firm-level growth dynamics. This complicates numerics. Methods for handling fat tails numerically have been proposed by Gouin-Bonenfant and Toda (2019) in the context of the wealth distribution and similar ideas should be applicable here.

Figure 1 shows a histogram of the normalized firm size distribution generated by the model in the setting of Example 3.2, with the equilibrium computed according to (S1)–(S5). Firm size is measured by log output and the entry distribution is also assumed to be lognormal. While the distribution looks lognormal at first approximation, the right hand tail is too heavy. In fact the distribution strongly exhibits the characteristics of a Pareto tail, as shown by the rank-size plot in Figure 2 (which uses the same data). In the next section we prove that the distribution is Pareto-tailed under this specification (which matches Gibrat's law) and a large range of alternative specifications.

4. Pareto Tails

Next we turn to the tail properties of the equilibrium distribution of firms identified by Theorem 3.4. To be certain that this distribution exists, we impose the conditions of

[11]Stronger statements are true when Assumption 3.1 holds. We show in the proof of Proposition 3.5 that when Assumption 3.1 is in force, the transition kernel Π_p is V-uniformly ergodic (Meyn and Tweedie, 2012, Chapter 16) for all $p > 0$, implying that the marginal distributions generated by Π_p converge to its unique stationary distribution at a geometric rate and yielding a range of sample path properties.

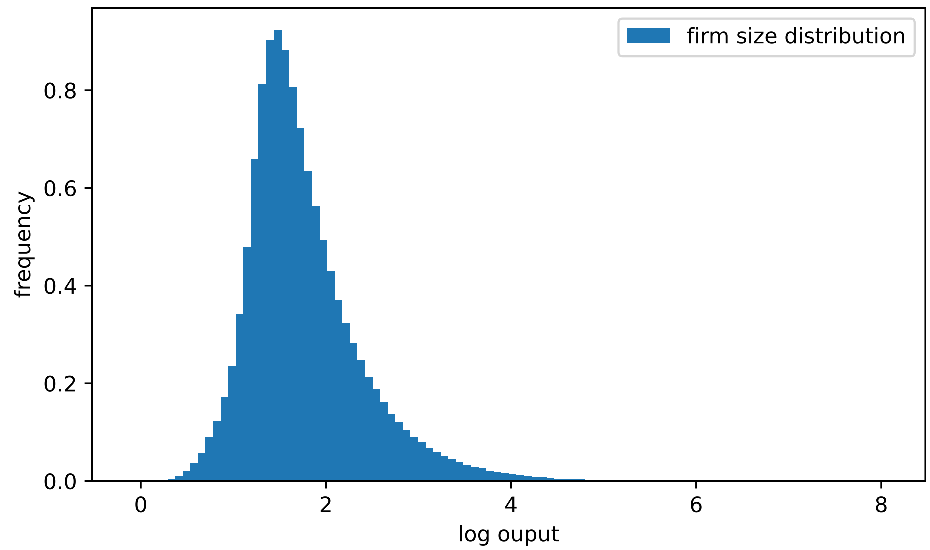

FIGURE 1. Histogram of the simulated log firm size distribution

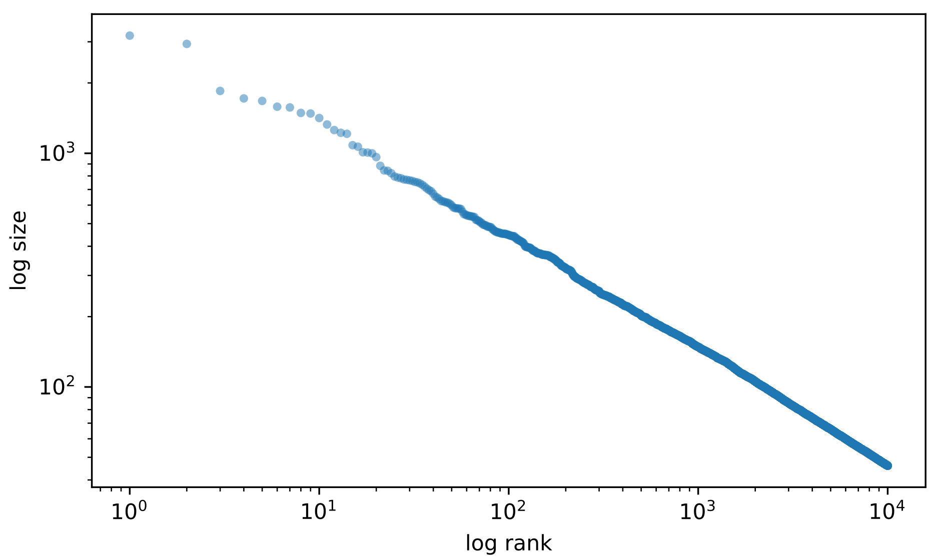

FIGURE 2. Rank-size plot of the simulated firm size distribution

Proposition 3.5. While we focus on productivity when analyzing firm size, heavy tails in productivity is typically mirrored or accentuated in profit-maximizing output.[12]

[12]For example, in the Cobb–Douglas case studied in Example 3.2, profit-maximizing output is convex in productivity.

It is convenient to introduce a function G and an IID sequence $\{W_t\}$ such that

$$\varphi_{t+1} = G(\varphi_t, W_{t+1}) \tag{19}$$

obeys the incumbent dynamics embodied in the Markov kernel Γ.[13] Such a representation can always be constructed (see, e.g., Bhattacharya and Majumdar (2007)). Let X be a random variable with distribution μ_{p^*}, where μ_{p^*} is the unique probability measure obeying (17) at the equilibrium entry price p^*. The firm size distribution[14] has a Pareto tail with tail index $\alpha > 0$ if there exists a $C > 0$ with

$$\lim_{x \to \infty} x^\alpha \, \mathbb{P}\{X > x\} = C. \tag{20}$$

In other words, the distribution is such that $\mathbb{P}\{X > x\}$ goes to zero like $x^{-\alpha}$. To investigate when X has this property, we impose the following restriction on the law of motion for incumbent firms. In stating it, we take W as a random variable with the same distribution as each W_t.

Assumption 4.1. There exists an $\alpha > 0$ and an independent random variable A with continuous distribution function such that $\mathbb{E}A^\alpha = 1$, the moments $\mathbb{E}A^{\alpha+1}$ and $\int z^\alpha \gamma(\mathrm{d}z)$ are both finite, and

$$\mathbb{E}\,|G(X, W)^\alpha - (AX)^\alpha| < \infty. \tag{21}$$

Condition (21) bounds the deviation between the law of motion (19) for incumbent productivity and Gibrat's law, which is where productivity updates via $\varphi_{t+1} = A_{t+1}\varphi_t$. The existence of a positive α such that $\mathbb{E}A^\alpha = 1$ requires that A puts at least some probability mass above 1. In terms of Gibrat's law $\varphi_{t+1} = A_{t+1}\varphi_t$, this corresponds to the natural assumption that incumbent firms grow with positive probability.

Theorem 4.1. *If Assumption 4.1 holds for some $\alpha > 0$, then the endogenous stationary distribution for firm productivity is Pareto-tailed, with tail index equal to α.*

While Assumption 4.1 involves X, which is endogenous, we can obtain it from various sufficient conditions that involve only primitives. For example, suppose there exist independent nonnegative random variables A and Y such that

---

[13] In other words, $\mathbb{P}\{G(\varphi, W_{t+1}) \in B\} = \Gamma(\varphi, B)$ for all $\varphi \geqslant 0$, $B \in \mathscr{B}$.

[14] In referring to this distribution, we ignore the distinction between the probability distribution μ_{p^*}, from which X is drawn, and the equilibrium firm size distribution μ^*, since one is a rescaled version of the other and hence the tail properties are unchanged.

(P1) Y has finite moments of all orders,

(P2) A satisfies the conditions in Assumption 4.1 for some $\alpha \in (0, 2)$, and

(P3) the bound $|G(\varphi, W) - A\varphi| \leqslant Y$ holds for all $\varphi \geqslant 0$.

We also assume that the first moment of γ is finite, although this is almost always implied by Assumption 2.4 (see, e.g., Examples 3.1–3.2).

Condition (P3) provides a connection between incumbent dynamics and Gibrat's law. Note that the dynamics in G can be nonlinear and, since Y is allowed to be unbounded, infinitely large deviations from Gibrat's law are permitted. One simple specification satisfying (P3) is when $G(\varphi, W) = A\varphi + Y$, which already replicates some empirically relevant properties (e.g., small firms exhibit more volatile and faster growth rates than large ones).

Conditions (P1)–(P3) only restrict incumbent dynamics (encapsulated by Γ in the notation of Sections 2–3). Since, in Theorem 4.1, the tail index is determined by α, these dynamics are the only primitive that influences the index on the Pareto tail. The range of values for α in (P2) covers standard estimates (see, e.g., Gabaix (2016)).

To show that (P1)–(P3) imply the conditions of Assumption 4.1, we proceed as follows. As A satisfies the conditions of Assumption 4.1, we only need to check that (21) holds. In doing so, we will make use of the elementary bound

$$|x^a - y^a| \leqslant \begin{cases} |x-y|^a & \text{if } 0 < \alpha \leqslant 1; \\ \alpha|x-y| \max\{x,y\}^{\alpha-1} & \text{if } 1 < \alpha \end{cases} \tag{22}$$

for nonnegative x, y. In the case $0 < \alpha \leqslant 1$, we therefore have, by (P3),

$$|G(X,W)^\alpha - (AX)^\alpha| \leqslant |G(X,W) - (AX)|^\alpha \leqslant Y^\alpha.$$

But Y has finite moments of all orders by (P1), so the bound in (21) holds.

Next consider the case $1 < \alpha < 2$. Using (22) again, we have

$$|G(X,W)^\alpha - (AX)^\alpha| \leqslant \alpha\, |G(X,W) - (AX)| \max\{G(X,W), AX\}^{\alpha-1}.$$

In view of (P3) above and the identity $2\max\{x,y\} = |x-y| + x + y$, we obtain

$$|G(X,W)^\alpha - (AX)^\alpha| \leqslant \alpha Y \left[Y + G(X,W) + (AX)\right]^{\alpha-1}.$$

Setting $a := 1/(\alpha - 1)$ and using Jensen's inequality combined with the fact that $\alpha < 2$ now yields

$$\mathbb{E}\,|G(X,W)^\alpha - (AX)^\alpha| \leqslant \alpha \left[\mathbb{E}Y^{a+1} + \mathbb{E}Y^a G(X,W) + \mathbb{E}Y^a(AX)\right]^{\alpha-1}.$$

We need to bound the three expectations on the right hand side. In doing so we use Lemma A.5 in the appendix, which shows that $\mathbb{E}X < \infty$ when $1 < \alpha < 2$.

The first expectations is finite by (P1). The third is finite by (P1) and independence of Y, A and X.[15] For the second, since Y is independent of X and W, finiteness of the expectation reduces to finiteness of $\mathbb{E}G(X,W)$. We have

$$G(X,W) = G(X,W)\mathbb{1}\{X < \bar{\varphi}(p^*)\} + G(X,W)\mathbb{1}\{X \geqslant \bar{\varphi}(p^*)\}.$$

Taking expectations and observing that, given $X \geqslant \bar{\varphi}(p^*)$, the random variable $G(X,W)$ has distribution $\Pi_{p^*}(X, \mathrm{d}\varphi')$, we have

$$\mathbb{E}G(X,W) \leqslant \int z\gamma(\mathrm{d}z) + \int\int \varphi'\,\Pi_{p^*}(\varphi, \mathrm{d}\varphi')\mu_{p^*}(\mathrm{d}\varphi) = \int z\gamma(\mathrm{d}z) + \int \varphi\mu_{p^*}(\mathrm{d}\varphi).$$

The equality on the right is due to stationarity of μ_{p^*} under the endogenous law of motion for firm productivity. Since $\int z\gamma(\mathrm{d}z)$ is finite by assumption and $\int \varphi\mu_{p^*}(\mathrm{d}\varphi) = \mathbb{E}X$, which is finite as stated above, we conclude that under (P1)–(P3), the conditions of Theorem 4.1 are satisfied.

5. Conclusion

In this paper we investigated the entry-exit model of Hopenhayn (1992) after removing the upper bound on firm productivity, allowing us to consider more realistic representations of firm growth. In this setting we provided an exact characterization of existence of stationary equilibria as well as a Lyapunov-type sufficient condition for existence. We also provided a new decomposition of the equilibrium distribution of firms, as well as new representations of the rate of entry and aggregate supply.

We showed that, when the boundedness assumption on productivity is removed, the Pareto tail of the distribution is predicted under a wide and empirically plausible class of specifications for firm-level productivity growth. Thus, by relaxing a purely technical assumption, we show that the Pareto tail in the firm size distribution is, in fact, a highly robust prediction of the Hopenhayn entry-exit model.

[15] Note that $\mathbb{E}A^\alpha = 1$ and, in the present case, we have $1 < \alpha < 2$, so finiteness of $\mathbb{E}A$ is assured.

The machinery employed in this paper to prove power law results draws on Goldie (1991), which uses implicit renewal theory to analyze Pareto tails of a range of time-invariant probability laws. The tool set recently developed in Beare and Toda (2022) is also well-suited to the setting we consider, and its application might lead to further insights.

The methodology developed above can potentially be applied to other settings where a power law is observed. For example, the wealth distribution is Pareto tailed, while the rate of return on wealth (and hence the growth rate of wealth) has been found to vary with the level of wealth in systematic ways (see, e.g., Fagereng et al. (2016)). Similarly, the distribution of city sizes tends to a Pareto tail. At the same time, Gibrat's law fails in this setting too (see, e.g., Córdoba (2008)). Such topics are left to future work.

Appendix A. Proofs

In the proofs we use the operator notation

$$(\Gamma u)(\varphi, p) := \int u(\varphi', p)\Gamma(\varphi, \mathrm{d}\varphi') \text{ for each } u \in \mathscr{C},$$

while $\mathscr{P}$ denotes the Borel probability measures on $\mathbb{R}_+$. The symbol $\preceq$ represents first order stochastic dominance. All undefined notation and terminology associated with Markov models follows Meyn and Tweedie (2012).

Throughout the appendix, all assumptions in Section 2 are in force.

A.1. Preliminary Results.

This section contains proofs of preliminary results needed for the main theorem. In the lemma below, we take $\delta \in (\beta, 1)$ from Assumption 2.7.

Lemma A.1. *The operator Γ is invariant on $\mathscr{C}$ and $\Gamma\kappa \leqslant \kappa/\delta$.*

Proof. The last claim is easy to check, since, by the definition of κ in (3), we have

$$\Gamma\kappa = \sum_{t\geqslant 0} \delta^t \Gamma^{t+1}\hat{\pi} = (1/\delta)\sum_{t\geqslant 0} \delta^{t+1}\Gamma^{t+1}\hat{\pi} \leqslant (1/\delta)\sum_{t\geqslant 0} \delta^t \Gamma^t \hat{\pi} = (1/\delta)\kappa. \tag{23}$$

Now fix $u \in \mathscr{C}$. That Γu is κ-bounded follows from the previous inequality and the pointwise bound $|u| \leqslant \|u\|_\kappa \kappa$. Continuity of Γu is immediate from Assumption 2.8. Regarding monotonicity, let $u_n = u\mathbb{1}\{u \leqslant n\} + n\mathbb{1}\{u > n\}$ for each $n \in \mathbb{N}$. Then u_n is increasing for each n and also bounded, so Γu_n is increasing for each n.[16] Moreover, by the Monotone Convergence Theorem, $\Gamma u_n \uparrow \Gamma u$. Since monotonicity is preserved under pointwise limits, Γu is also increasing. Hence $\Gamma u \in \mathscr{C}$ as claimed. □

Proof of Lemma 3.1. Pick any $u \in \mathscr{C}$. Using (23) and $\hat{\pi} \leqslant \kappa$, we have

$$|Tu| = |\pi + \beta \max\{0, \Gamma u\}| \leqslant \hat{\pi} + \beta\Gamma|u| \leqslant \hat{\pi} + \beta\|u\|_\kappa \Gamma\kappa \leqslant (1 + \beta\|u\|_\kappa/\delta)\kappa.$$

Hence $\|Tu\|_\kappa$ is finite. In addition, Tu is continuous and increasing because $Tu = \pi + \beta \max\{0, \Gamma u\}$ and π and Γu both have these properties (by Assumptions 2.2 and 2.8). Hence T maps $\mathscr{C}$ into itself. In addition, T is a contraction mapping, since, given u, v in $\mathscr{C}$,

$$|Tu - Tv| \leqslant \beta\Gamma|u - v| \leqslant \beta\|u - v\|_\kappa \Gamma\kappa \leqslant (\beta/\delta)\|u - v\|_\kappa \kappa.$$

Dividing both sides by κ and taking the supremum yields $d(Tu, Tv) \leqslant (\beta/\delta) d(u, v)$. Recalling that $\delta > \beta$, the claim of contractivity is established.

To see that the fixed point $\bar{v}$ is strictly increasing, pick any $w \in \mathscr{C}$ and observe that $Tw = \pi + \beta \max\{0, \Gamma w\}$ is strictly increasing, since Γw is increasing and π is strictly increasing. In other words, T maps elements of $\mathscr{C}$ into strictly increasing functions. Given that $\bar{v} = T\bar{v}$, the function $\bar{v}$ must itself have these properties.

Finally, to see that $\bar{v}(\varphi, p) < 0$ if $\varphi = 0$ or $p = 0$, let $h(\varphi, p) := \sum_{t \geqslant 1} \beta^t \mathbb{E}_\varphi |\pi(\varphi_t, p)|$ where $\{\varphi_t\}$ is a productivity process starting at φ and generated by Γ. Clearly $\bar{v}(\varphi, p) \leqslant \pi(\varphi, p) + h(\varphi, p)$. If $p = 0$, then $\pi(\varphi, p) < 0$ and $h(\varphi, p) \leqslant 0$ by Assumption 2.2. Hence $\bar{v}(\varphi, p) < 0$. In addition, if $\varphi = 0$, then profits are negative in the first period, by Assumption 2.2, and subsequent lifetime profits are nonpositive by Assumption 2.6. Once again, we have $\bar{v}(\varphi, p) < 0$. □

Proof of Lemma 3.2. Let $\Phi(p) := \{\varphi \geqslant 0 \mid (\Gamma\bar{v})(\varphi, p) \geqslant 0\}$. This set is nonempty when $p > 0$ by $\bar{v} \geqslant \pi$ and Assumption 2.3. Moreover, $\Phi(p)$ is closed because, if $\{\varphi_n\} \subset \Phi(p)$ and $\varphi_n \to \varphi$, then, by the continuity in Assumption 2.8, we have $0 \leqslant (\Gamma\bar{v})(\varphi_n, p) \to (\Gamma\bar{v})(\varphi, p)$. Hence $(\Gamma\bar{v})(\varphi, p) \geqslant 0$ and, therefore, $\varphi \in \Phi(p)$. Since $\Phi(p)$ is closed and nonempty when $p > 0$, $\bar{\varphi}(p) = \min \Phi(p)$ exists in $\mathbb{R}_+$.

Due to monotonicity of $\bar{v}$, the correspondence Φ is such that $p \leqslant q$ implies $\Phi(p) \subset \Phi(q)$. Hence $\bar{\varphi}(p) = \min \Phi(p)$ is decreasing. Moreover, the set $\Phi(p)$ does not contain

[16] By Assumption 2.3, the kernel Γ is monotone increasing. This implies that Γf is increasing whenever f is measurable, increasing and bounded.

0 because, for any $p \geqslant 0$, we have $\Gamma\bar{v}(0,p) = \bar{v}(0,p) < 0$, where the equality is by Assumption 2.3 and the inequality is by Lemma 3.1. Hence $\bar{\varphi}(p) > 0$. □

Proof of Lemma 3.3. Let $e(p) := \int \bar{v}(\varphi,p)\gamma(\mathrm{d}\varphi) - c_e$. The function e is finite on $\mathbb{R}_+$ because $\bar{v} \leqslant \kappa$ and

$$\int \kappa(\varphi,p)\gamma(\mathrm{d}\varphi) = \sum_{t\geqslant 0} \delta^t \int [\pi(\varphi,p)+b]\,\gamma(\mathrm{d}\varphi) < \infty$$

by Assumption 2.7. The function e is also continuous on $\mathbb{R}_+$. To see this, take $p_n \to p$. Since convergent sequences are bounded, we can choose $\bar{p}$ such that $p_n \leqslant \bar{p}$ for all n. By monotonicity, it follows that $\bar{v}(\varphi',p_n) \leqslant \bar{v}(\varphi',\bar{p})$ for all φ'. Continuity of $\bar{v}$ and the Dominated Convergence Theorem now give $e(p_n) \to e(p)$.

If $p = 0$, then, by Lemma 3.1, we have $\bar{v}(\varphi,p) < 0$ for all φ, so $e(p) < 0$. Conversely, if p is large enough, then $\int \pi(\varphi,p)\gamma(\mathrm{d}\varphi) \geqslant c_e$ by Assumption 2.5. As $\bar{v} \geqslant \pi$, this implies that $e(p) \geqslant 0$. Hence, by the Intermediate Value Theorem, there exists a $p > 0$ such that $e(p) = 0$. Uniqueness now follows from strict monotonicity of e, which in turn rests on strict monotonicity of $\bar{v}$ (see Lemma 3.1). □

Lemma A.2. *For all $p > 0$, the transition kernel Π_p defined in (18) is aperiodic, γ-irreducible and admits the accessible atom $\mathbf{a}_p := [0,\bar{\varphi}(p))$.*

Proof. Fix $\varphi \in \mathbb{R}_+$ and let B be any Borel set such that $\gamma(B) > 0$. Let $\{\varphi_t\}$ be a Markov process on $\mathbb{R}_+$ generated by Π_p and starting at $\varphi_0 = \varphi$. Evidently, if $\varphi < \bar{\varphi}(p)$, then $\Pi_p(\varphi,B) = \gamma(B) > 0$, so B is reachable from φ. If instead $\varphi \geqslant \bar{\varphi}(p)$, then we let m be the smallest $n \in \mathbb{N}$ such that $\Gamma^n(\varphi,[0,\bar{\varphi}(p)) > 0$. By the Chapman–Kolmogorov equations, we have

$$\mathbb{P}\{\varphi_{m+1} \in B\} = \int \Pi_p(\varphi',B)\Pi_p^m(\varphi,\mathrm{d}\varphi') \geqslant \int_0^{\bar{\varphi}(p)} \Pi_p(\varphi',B)\Pi_p^m(\varphi,\mathrm{d}\varphi').$$

The right-hand side evaluates to $\gamma(B)\,\Gamma^m(\varphi,[0,\bar{\varphi}(p))$, which is strictly positive by the assumed positivity of $\gamma(B)$ and the definition of m. Again B is reachable, and hence Π_p is γ-irreducible. Aperiodicity now follows from Assumption 2.4.

Finally, $\gamma(\mathbf{a}_p) > 0$ by Assumption 2.4. The interval $\mathbf{a}_p$ is an atom because $\Pi_p(\varphi,A) = \Pi_p(\psi,A) = \gamma(A)$ for all $\varphi,\psi < \bar{\varphi}(p)$. □

The next lemma discusses petite sets, as defined in Meyn and Tweedie (2012).

Lemma A.3. *If $p > 0$ and $d > 0$, then $[0, d]$ is a petite set for Π_p.*

Proof. Fix $p, d > 0$. It suffices to show existence of a nontrivial Borel measure ν on $\mathbb{R}_+$ and an $m \in \mathbb{N}$ such that $\Pi_p^m(\varphi, B) \geqslant \nu(B)$ whenever $0 \leqslant \varphi \leqslant d$ and $B \in \mathscr{B}$. Let $\mathbf{a}_p := [0, \bar{\varphi}(p))$ and take the smallest $n \in \mathbb{N}$ such that $\varepsilon := \Gamma^n(d, \mathbf{a}_p) > 0$. (This n is finite by Assumption 2.3.) Pick any $\varphi \in [0, d]$ and $B \in \mathscr{B}$, and let $\{\varphi_t\}$ be generated by Π_p from initial condition φ. By the law of total probability,

$$\Pi_p^{n+1}(\varphi, B) = \mathbb{P}\{\varphi_{n+1} \in B\} \geqslant \mathbb{P}\{\varphi_{n+1} \in B \mid \varphi_n \in \mathbf{a}_p\}\mathbb{P}\{\varphi_n \in \mathbf{a}_p\}.$$

By monotonicity of Γ and the definition of Π_p, we then have

$$\Pi_p^{n+1}(\varphi, B) \geqslant \gamma(B)\mathbb{P}\{\varphi_n \in \mathbf{a}_p\} = \gamma(B)\Gamma^n(\varphi, \mathbf{a}_p) \geqslant \gamma(B)\Gamma^n(d, \mathbf{a}_p) = \gamma(B)\varepsilon.$$

Setting $\nu := \varepsilon\gamma$ and $m := n + 1$ therefore gives $\Pi_p^m(\varphi, B) \geqslant \nu(B)$, which verifies the claim in the lemma. □

Lemma A.4. *Fix $p > 0$ and let Π_p and $\mathbf{a}_p$ be as in Lemma A.2. The following statements are equivalent:*

(i) *There exists a $\mu_p \in \mathscr{P}$ such that (17) holds.*
(ii) *Expected firm lifespan $\mathbb{E}\tau(p)$ is finite.*

If either and hence both of these conditions holds, then $\mu_p(\mathbf{a}_p) > 0$ and

$$\mu_p(B) = \mu_p(\mathbf{a}_p) \cdot \mathbb{E} \sum_{t=1}^{\tau(p)} \mathbb{1}\{\varphi_t \in B\} \tag{24}$$

for all $B \in \mathscr{B}$. In addition, aggregate supply obeys

$$0 < \int q(\varphi, p)\mu_p(\mathrm{d}\varphi) = \mu_p(\mathbf{a}_p)\mathbb{E}\,\ell(p). \tag{25}$$

Proof of Lemma A.4. Fix $p > 0$. The kernel Π_p is γ-irreducible by Lemma A.2. The set $\mathbf{a}_p = [0, \bar{\varphi}(p))$ is an atom for Π_p because $\Pi_p(\varphi, B) = \gamma(B)$ whenever $\varphi \in \mathbf{a}_p$. Moreover, $\gamma(\mathbf{a}_p) > 0$ by Assumption 2.4. It now follows from Theorem 10.2.2 of Meyn and Tweedie (2012) that Π_p is positive recurrent – and hence admits a stationary probability μ_p – if and only the expected return time to $\mathbf{a}_p$ is finite. This is equivalent to $\mathbb{E}\tau(p) < \infty$, which proves the first claim in the lemma.

Now suppose that $\mathbb{E}\tau(p) < \infty$ holds and let μ_p be stationary for Π_p. Equation (24) follows from See Theorem 10.4.9 of Meyn and Tweedie (2012). Positivity of $\mu_p(\mathbf{a}_p)$

holds because $\mathbf{a}_p$ is an accessible atom (Lemma A.2). Since output q is nonnegative, (24) extends via the Monotone Convergence Theorem to

$$\int q(\varphi, p)\mu_p(\mathrm{d}\varphi) = \mu_p(\mathbf{a}_p) \cdot \mathbb{E} \sum_{t=1}^{\tau(p)} q(\varphi_t, p) = \mu_p(\mathbf{a}_p)\mathbb{E}\, \ell(p),$$

which gives the equality in (25). Positivity of supply follows from $q(\varphi, p) > 0$ for all $\varphi > 0$ and $\mu_p(\mathbf{a}_p) > 0$, since

$$\int q(\varphi, p)\mu_p(\mathrm{d}\varphi) \geqslant \int_{\mathbf{a}_p} q(\varphi, p)\mu_p(\mathrm{d}\varphi). \qquad \square$$

A.2. **Existence and Uniqueness.** In what follows, p^* is the equilibrium entry price, as defined in (12).

Proof of Theorem 3.4. Suppose first that $\mathbb{E}\, \ell(p^*) = \infty$ and yet there exists a pair (M^*, μ^*) such that (p^*, M^*, μ^*) is stationary recursive equilibrium (SRE). By definition, μ^* is a stationary productivity measure and a finite invariant measure for Π_{p^*}. This means that (i) in Lemma A.4 holds. Hence we can apply (25), obtaining

$$\int q(\varphi, p^*)\mu^*(\mathrm{d}\varphi) = \mu_p(\mathbf{a}_p)\mathbb{E}\, \ell(p^*) = \infty.$$

Since $\mu_p(\mathbf{a}_p) > 0$, aggregate supply is infinite, while demand $D(p^*)$ is finite. Contradiction.

Now suppose instead that $\mathbb{E}\, \ell(p^*)$ is finite. Observe that, by monotonicity of q and the definition of $\tau(p^*)$, we have

$$\sum_{t=1}^{\tau(p^*)} q(\varphi_t, p^*) \geqslant \sum_{t=1}^{\tau(p^*)} q(\bar{\varphi}(p^*), p^*) = \tau(p^*)q(\bar{\varphi}(p^*), p^*).$$

$$\therefore \quad \tau(p^*) \leqslant c \sum_{t=1}^{\tau(p^*)} q(\varphi_t, p^*) \text{ for some } c > 0.$$

Taking expectations gives $\mathbb{E}\tau(p^*) \leqslant \mathbb{E}\, \ell(p^*) < \infty$. Hence, by Lemma A.4, the distribution μ_{p^*} satisfying (17) at p^* is well defined, and (24) and (25) both hold at p^*. We take μ^* as given by step (S4), which is well defined by (25), and M^* as given by (S5); that is $M^* := \mu^*\{\varphi : \varphi < \bar{\varphi}(p^*)\}$.

By construction, the triple (p^*, M^*, μ^*) satisfies all of the conditions in Section 3.2. For example, the goods market clears because, by the definition of the scaling constant s

in (S4), we have $\int q(\varphi, p^*)\mu^*(\mathrm{d}\varphi) = s \int q(\varphi, p^*)\mu(\mathrm{d}\varphi) = D(p^*)$. The time-invariance condition 7 holds because, given any B in $\mathscr{B}$,

$$\begin{aligned} \int \Gamma(\varphi, B)\mathbb{1}\{\varphi \geqslant \bar{\varphi}(p^*)\}\mu^*(\mathrm{d}\varphi) + M^*\gamma(B) \\ = \int \Gamma(\varphi, B)\mathbb{1}\{\varphi \geqslant \varphi^*\}\mu^*(\mathrm{d}\varphi) + \mu^*\{\varphi < \varphi^*\}\gamma(B) \\ = s \int \left[\Gamma(\varphi, B)\mathbb{1}\{\varphi \geqslant \varphi^*\} + \mathbb{1}\{\varphi < \varphi^*\}\gamma(B)\right] \mu(\mathrm{d}\varphi). \end{aligned}$$

Since μ satisfies (17), this last expression is just $s\mu(B)$, or, equivalently, $\mu^*(B)$. Hence (p^*, M^*, μ^*) is an SRE.

The triple (p^*, M^*, μ^*) is the only SRE in $\mathcal{E}$ because the time invariance condition has at most one solution, by Lemma A.2, and the price p^* is uniquely determined by Lemma 3.3. Given p^* and μ^*, the constant M^* is then uniquely determined by (S5).

The decomposition (13) follows from (24). Multiplying both sides by s from (S4) gives

$$\mu^*(B) = \mu^*(\mathbf{a}) \cdot \mathbb{E} \sum_{t=1}^{\tau} (p^*)\mathbb{1}\{\varphi_t \in B\} = M^* \cdot \mathbb{E} \sum_{t=1}^{\tau} (p^*)\mathbb{1}\{\varphi_t \in B\}.$$

Equation (14) follows from (25) and the definition of M^*. □

A.3. **Drift.**

Proof of Proposition 3.5. Adopting the conditions of Proposition 3.5, we first show that Π_p is V-uniformly ergodic for all $p > 0$ via Theorem 16.1.2 of Meyn and Tweedie (2012). Lemma A.2 shows that Π_p is irreducible and aperiodic, so we need only show that the drift condition (V4) defined in Chapter 15 of Meyn and Tweedie (2012) is holds with respect to a petite set. By Lemma 15.2.8 of the same reference, this will be true whenever there exists a nonnegative function V on $\mathbb{R}_+$ such that the sublevel set $C_a := \{\varphi \in \mathbb{R}_+ : V(\varphi) \leqslant a\}$ is petite for each $a \geqslant 0$ and, for some positive constants $\alpha < 1$ and $K < \infty$,

$$\int V(\varphi')\Pi_p(\varphi, \mathrm{d}\varphi') \leqslant \alpha V(\varphi) + K \text{ for all } \varphi \geqslant 0. \tag{26}$$

Set $V(\varphi) = q(\varphi, p)$. For this function the sublevel sets C_a are all intervals of the form $[0, d]$ for some $d \geqslant 0$ due to monotonicity of q. Such sets are petite by Lemma A.3.

Moreover, for any fixed $\varphi \geqslant 0$, the definition of Π_p and the drift condition for incumbents in (15) yields

$$\int V(\varphi')\Pi_p(\varphi, \mathrm{d}\varphi') \leqslant \int V(\varphi')\gamma(\mathrm{d}\varphi') + \int V(\varphi')\Gamma(\varphi, \mathrm{d}\varphi').$$

The first term is finite by Assumption 2.4. The second term is bounded by (15). Putting these bounds together yields (26) with $\alpha := \lambda$ and $K := \int V(\varphi')\gamma(\mathrm{d}\varphi') + L$.

Next we claim that $\mathbb{E}\,\ell(p)$ is finite for this same arbitrary p. To see this, let μ be the unique stationary distribution of Π_p, existence of which is guaranteed by V-uniform ergodicity (see Theorem 16.1.2 of Meyn and Tweedie (2012)). The term $m(p) := \int q(\varphi, p)\mu(\mathrm{d}p)$ is finite by Proposition 4.24 of Hairer (2018), combined with the fact that $V(\varphi) := q(\varphi, p)$ satisfies the drift condition (26). Moreover, μ satisfies (24), which can be expressed as $\mu(B) = c\,\mathbb{E}\sum_{t=1}^{\tau(p)} \mathbb{1}_B\{\varphi_t\}$ for some positive constant c. We can extend up from indicator functions such as $\mathbb{1}_B$ to the nonnegative measurable function V via the Monotone Convergence Theorem, so

$$\int V(\varphi)\mu(\mathrm{d}\varphi) = c\,\mathbb{E}\sum_{t=1}^{\tau(p)} V(\varphi_t) = c\,\mathbb{E}\sum_{t=1}^{\tau(p)} q(\varphi_t, p) = c\,\cdot\mathbb{E}\,\ell(p).$$

Hence $m(p) := \int q(\varphi, p)\mu(\mathrm{d}p) = \int V(\varphi)\mu(\mathrm{d}p) = c\,\mathbb{E}\,\ell(p)$. Since $m(p)$ was just shown to be finite and c is positive, the claim is verified.

As $\mathbb{E}\,\ell(p)$ is finite for all positive p, it is finite at p^*, and hence the conclusions of Theorem 3.4 hold. □

A.4. **Pareto Tails.**

Proof of Theorem 4.1. To begin, recall from (19) that the recursion $\varphi_{t+1} = G(\varphi_t, W_{t+1})$ reproduces the Markov dynamics for an incumbent firm embodied in Γ. If we take $\{Z_t\}$ to be IID draws from γ and set

$$H(\varphi, w, z) := G(\varphi, w)\mathbb{1}\{\varphi \geqslant \bar{\varphi}(p^*)\} + z\mathbb{1}\{\varphi < \bar{\varphi}(p^*)\}, \tag{27}$$

then the recursion

$$\varphi_{t+1} = H(\varphi_t, W_{t+1}, Z_{t+1}) \tag{28}$$

reproduces equilibrium firm dynamics corresponding to Π_{p^*}.

Continuing to the proof of Theorem 4.1, we make use of the implicit renewal theory found in Corollary 2.4 of Goldie (1991). By Assumption 4.1, the distribution function of A is continuous and hence $\ln A$ is nonarithmetic. Moreover, $\mathbb{E}A^\alpha = 1$ and $\mathbb{E}A^{\alpha+1} < \infty$, the latter of which gives $\mathbb{E}A^\alpha \max\{\ln A, 0\} < \infty$. Hence A satisfies the conditions of Lemma 2.2 of Goldie (1991), and, as a result, we need only check that $\mathbb{E}|H(X,W,Z)^\alpha - (AX)^\alpha|$ is finite for H defined in (27).[17] For this it suffices to bound expectation of the two random variables

$$I_1 := |H(X,W,Z)^\alpha - (AX)^\alpha| \, \mathbb{1}\{X < \bar{\varphi}(p^*)\} = |Z^\alpha - (AX)^\alpha| \, \mathbb{1}\{X < \bar{\varphi}(p^*)\}$$

and

$$I_2 := |H(X,W,Z)^\alpha - (AX)^\alpha| \, \mathbb{1}\{X \geqslant \bar{\varphi}(p^*)\} \leqslant |G(X,W)^\alpha - (AX)^\alpha|$$

For I_1 we can use the triangle inequality and the bound on X to obtain $I_1 \leqslant Z^\alpha + A^\alpha \bar{\varphi}(p^*)$, and the expectation of the right hand side is finite by our assumptions on A and Z. For I_2, finiteness of expectation holds by the inequality on the right hand side of the definition of I_2 and Assumption 4.1. □

Lemma A.5. *Under conditions (a)–(c) of Section 4, the first moment of the firm size distribution is finite whenever $\alpha > 1$.*

Proof. Let $\{Z_t\}$ be IID draws from γ. Consider the upper bound process $U_{t+1} = A_{t+1}U_t + Y_{t+1} + Z_{t+1}$. This dominates the equilibrium process pointwise, as can be seen by comparing it with (27). It follows that the stationary μ of (27) is stochastically dominated by the stationary distribution of the upper bound process whenever the latter exists. Hence it suffices to show that the stationary solution to the upper bound process has finite first moment.

Since $\mathbb{E}A_{t+1}^\alpha = 1$ and $\alpha > 1$, we must have $\mathbb{E}A_{t+1} < 1$ (see, e.g., p. 48 of Buraczewski et al. (2016)). Finiteness of the first moment of the stationary solution to the upper bound process now follows from Theorem 5.1 of Vervaat (1979), provided that the additive component $Y_{t+1} + Z_{t+1}$ of this process has finite first moment. This is true under the stated assumptions, so the proof of Lemma A.5 is done. □

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[17]This will confirm Eq. (2.16) in Goldie (1991), which provides a Pareto law with index α for X.

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