Stationary Distributions in Monotone Markov Models

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Motivation

Distributions matter

- Distribution of wealth
- Distribution of income
- Distribution of consumption
- Distribution of money holdings
- Firm size distribution
- Measuring downside risk
- etc.

When modeling, most of our focus is on **stationary distributions**

- Also called ergodic / invariant distributions

But when do they exist?

More generally, for a given economic model, when do we have

- existence
- uniqueness
- stability (stationary distribution is attracting)
- ergodicity (time series averages → cross-sectional means)

Some Reminders

Consider

$$P = \begin{pmatrix} 0.97 & 0.02 & 0.01 \\ 0.03 & 0.95 & 0.02 \\ 0.04 & 0.06 & 0.90 \end{pmatrix}$$

Updating distributions:

$$\varphi_{t+1}(y) = \sum_x P(x,y)\varphi_t(x)$$

With φ_t and φ_{t+1} are row vectors,

$$\varphi_{t+1} = \varphi_t P$$

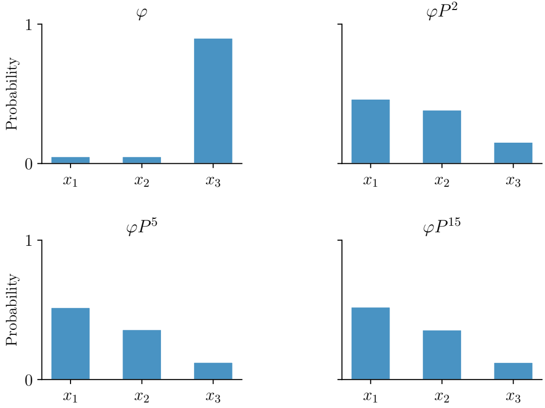


Figure: $\varphi_{t+1} = \varphi_t P$

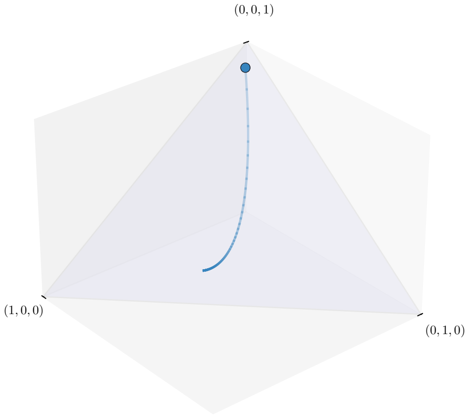


Figure: $\varphi_{t+1} = \varphi_t P$

Eg. For this choice of P, the distribution

$$\varphi^* = (0.52,\ 0.36,\ 0.12)$$

is **stationary**:

$$\varphi^* P = \varphi^*$$

Moreover, since P is everywhere positive, we have uniqueness and **global stability**:

$$\lim_{t \to \infty} \varphi P^t = \varphi^* \text{ for all } \varphi \in \mathscr{P}$$

(P everywhere positive $\implies$ aperiodic and irreducible)

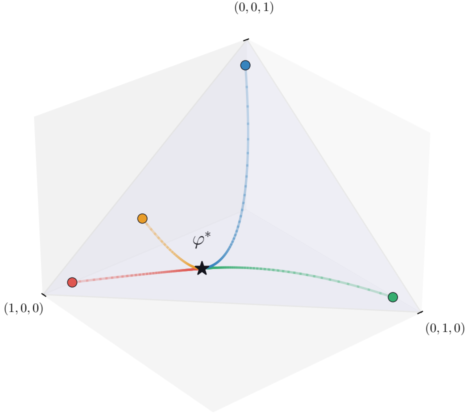


Figure: $\varphi P^t \to \varphi^*$ for all $\varphi \in \mathscr{P}$

Continuous State Case

Now P is now a **stochastic kernel**:

$$P(x, B) = \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$$

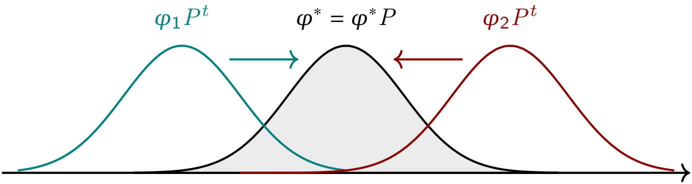
Updating distributions:

$$\varphi_{t+1}(y) = \int P(x, y)\varphi_t(\mathrm{d}x)$$

As before,

$$\varphi P^t = \text{distribution of } X_t \text{ when } X_0 \sim \varphi$$

When do we have existence / uniqueness / global stability?

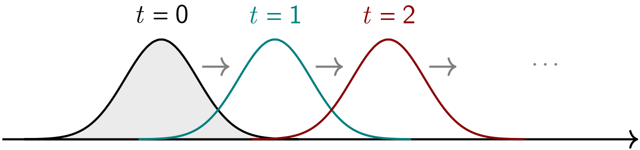


Analysis is a bit trickier...

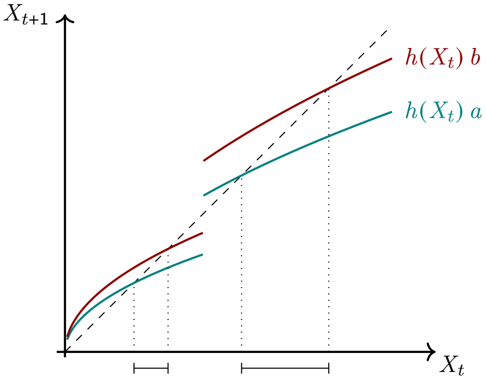
Failure of Existence

With general state space S, existence can easily fail

Eg. $X_{t+1} = X_t + 1$

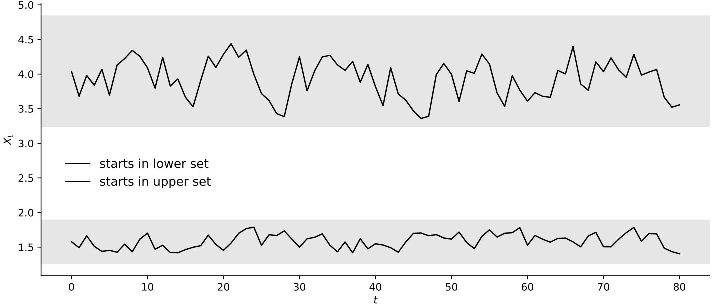


Failure of Uniqueness / Stability

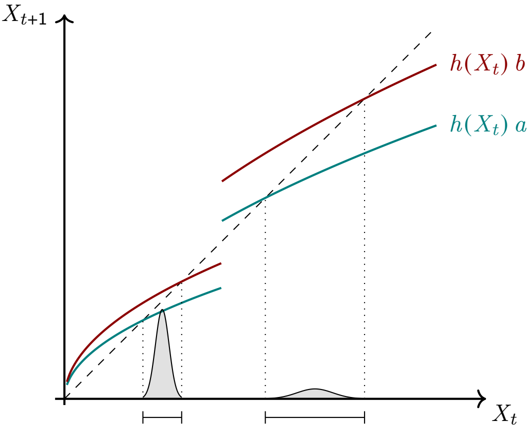


Eg. $X_{t+1} = h(X_t)\xi_{t+1}$ and $a \leqslant \xi_t \leqslant b$

Rich country and poor country



In this case mixing is insufficient and uniqueness / stability fails



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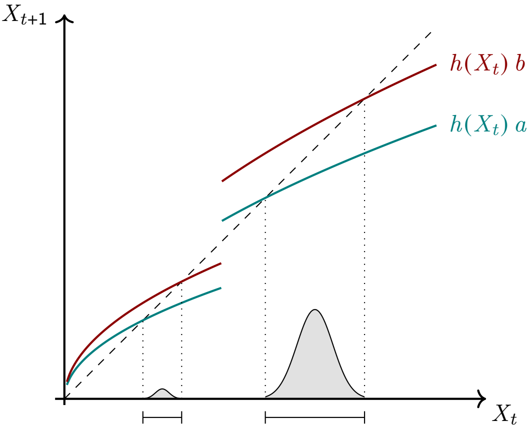


Summary

Global stability requires

1. **non-divergence**
2. **sufficient mixing**

Sufficient conditions are available (see, e.g., Meyn and Tweedie (2009))

- Eg. Non-divergence: Show that $\mathbb{E}|X_t|$ is bounded in t
- Eg. Mixing: hit every set of positive Lebesgue measure

But for economic models these conditions can be hard to prove...

Monotonicity

One side condition that can be helpful is monotonicity

- Relatively common in economic models
- When monotonicity holds, mixing conditions can be weaker

Stochastic kernel P on $(S, \preceq)$ is called **increasing** when

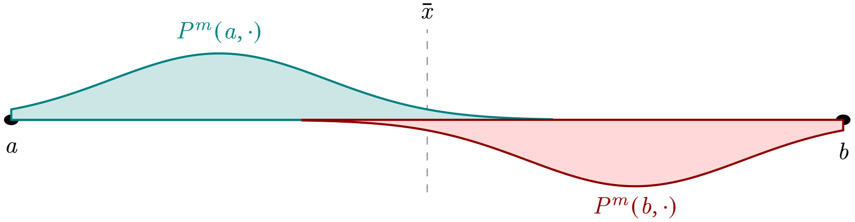
$$x \preceq x' \quad \implies \quad P(x, \cdot) \preceq_{\mathrm{sd}} P(x', \cdot)$$

Eg. Higher wealth today increases likelihood of high wealth tomorrow

Foundational Result: Hopenhayn and Prescott

A stochastic kernel P on $S = [a, b]$ is said to satisfy the **MMC** if $\exists\, \bar{x} \in S$ and $m \in \mathbb{N}$ such that

$$P^m(a, U_{\bar{x}}) > 0 \quad \text{and} \quad P^m(b, D_{\bar{x}}) > 0$$



Theorem: Hopenhayn and Prescott (1992)

Let

- $S = [a, b]$ and
- P be increasing

If P satisfies the MMC, then P is globally stable

Intuition:

- $S = [a, b]$ prevents divergence
- MMC supplies sufficent mixing

Widely cited:

- Chatterjee and Shukayev (2010)
- Hidalgo-Cabrillana (2009)
- Samaniego (2008)
- Marcet et al. (2007)
- Antunes and Cavalcanti (2007)
- Le Grand and Ragot (2022)
- Light and Weintraub (2022)
- Balbus et al. (2025)
- Kam et al. (2025)
- ...

Limitations

1. Assuming $S = [a, b]$ is not ideal
   - Standard cross-sectional distributions are Pareto tailed
   - Most standard parametric classes not compactly supported
2. Discrete time only — do ideas carry over to continuous time?
3. Sufficient but not necessary conditions

This Paper

We provide new results for such models

- Compact and noncompact state spaces
- Continuous and discrete time
- Necessary and sufficient conditions

Foundations: A new fixed point theorem for contractive maps

Applications: wage dynamics, belief shocks, income dynamics

Related literature

- Kamihigashi and Stachurski (2012)
- Kamihigashi and Stachurski (2014)
- Kamihigashi and Stachurski (2016)
- Foss et al. (2018)
- Kamihigashi and Stachurski (2019)
- Foss and Scheutzow (2024)
- Light (2024)
- Light (2026), etc., etc.

Dynamics

1. Discrete time: P is a fixed stochastic kernel and each trajectory is a sequence

$$t \mapsto \varphi P^t \qquad (t \in \mathbb{Z}_+)$$

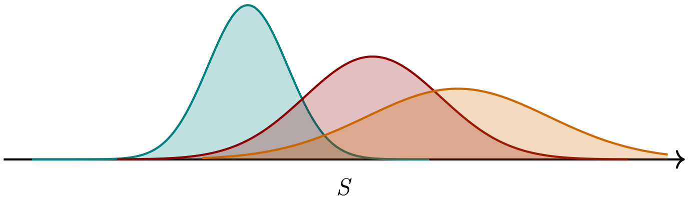
2. Continuous time: $(P_t)_{t \geqslant 0}$ is a transition probability function, with

$$P_t(x, B) = \mathbb{P}\{X_t \in B \mid X_0 = x\}$$

and each trajectory is a flow

$$t \mapsto \varphi P_t \qquad (t \in \mathbb{R}_+)$$

Let $\mathscr{P}(S)$ be the set of all distributions on S



- S is a metric space with partial order $\preceq$

State Space

Assumption 1

Subsets of S are compact if and only if they are closed and order bounded

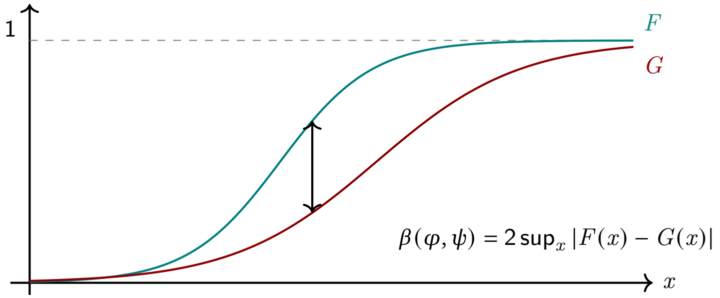
Egs.

- Hopenhayn and Prescott (1992), where $S = [a, b]$
- $\mathbb{R}^n$ with the usual metric and partial order $\leqslant$
- $\mathbb{R}^n_+$ with the usual metric and partial order $\leqslant$
- etc.

Measurement

For distance between $\varphi, \psi \in \mathscr{P}(S)$ we use the **Bhattacharya metric**

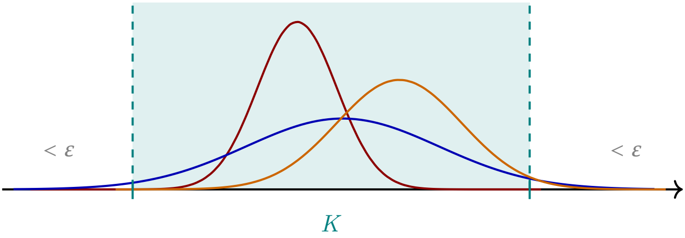
In one dimension, for the CDFs,



Tightness

We recall that a set $\mathscr{P}_0 \subset \mathscr{P}(S)$ is **tight** if probability mass doesn't escape to the edges

$$\forall\, \varepsilon > 0,\ \exists \text{ compact } K \subset S \ \text{ s.t. } \inf_{\varphi \in \mathscr{P}_0} \varphi(K) \geqslant 1 - \varepsilon$$



We say that $(P_t)_{t \geqslant 0}$ has a **tight trajectory** if

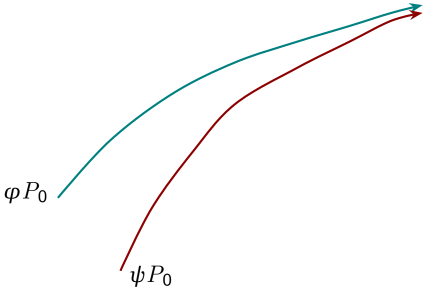
$$\{\varphi P_t : t \geqslant 0\} \text{ is tight for some } \varphi \in \mathscr{P}(S)$$

Egs.

- In Hopenhayn and Prescott (1992), every trajectory is tight
- If $\mathbb{E}|X_t|$ is bounded over t then φP_t is tight

We say that $(P_t)_{t \geqslant 0}$ is **asymptotically contractive** if

$$\lim_{t \to \infty} \beta(\varphi P_t, \psi P_t) = 0 \qquad \text{for all } \varphi, \psi \in \mathscr{P}(S)$$



Necessary and Sufficient Conditions

Theorem 1

If $(P_t)_{t \geqslant 0}$ is an increasing transition probability function on S, then

$$(P_t)_{t \geqslant 0} \text{ is globally stable}$$
$$\Longleftrightarrow$$
$$(P_t)_{t \geqslant 0} \text{ is asymptotically contractive and has a tight trajectory}$$

- Discrete time version is analogous
- Tightness arguments well-known — e.g. show $\mathbb{E}|X_t|$ bounded
- But how do we prove asymptotically contractive?

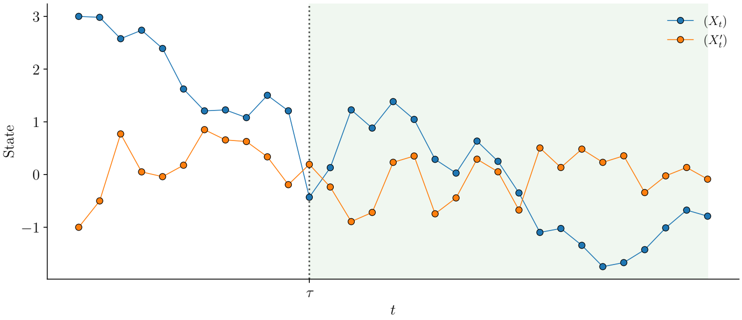
Order Mixing

Let $(P_t)_{t \geqslant 0}$ be an increasing transition probability function on S

We say that $(P_t)_{t \geqslant 0}$ is **weakly order mixing** if, given any X_0 and X_0', we can construct processes $(X_t)_{t \geqslant 0}$ and $(X_t')_{t \geqslant 0}$ such that

1. both processes are driven by $(P_t)_{t \geqslant 0}$, and
2. $X_t \preceq X_t'$ eventually with probability one

Let τ = first time $X_t \preceq X'_t$



Weak order mixing: for all X_0, X'_0, exist $(X_t)_{t\geqslant 0}$ and $(X'_t)_{t\geqslant 0}$ such that

$$\mathbb{P}\{\tau < \infty\} = 1$$

Theorem 2: Order-Theoretic Coupling Inequality

If $(P_t)_{t \geqslant 0}$ is increasing with $X_0 \sim \varphi$ and $X_0' \sim \psi$, then

$$\beta(\varphi P_t, \psi P_t) \leqslant 2\mathbb{P}\{\tau > t\} \quad \text{for all } t \geqslant 0$$

Hence

$$\mathbb{P}\{\tau < \infty\} = 1 \implies \lim_{t \to \infty} \mathbb{P}\{\tau > t\} = 0$$
$$\implies \lim_{t \to \infty} \beta(\varphi P_t, \psi P_t) = 0$$

In other words,

$$\text{weak order mixing} \implies \text{asymptotically contractive wrt } \beta$$

Corollary 1

An increasing $(P_t)_{t \geqslant 0}$ on S is globally stable whenever

1. $(P_t)_{t \geqslant 0}$ is weakly order mixing and
2. $(P_t)_{t \geqslant 0}$ has at least one tight trajectory

- Global stability $\iff$ sufficient mixing + non-divergence
- Analogous result holds in discrete time as well

Application 1: H & P in Continuous Time

Theorem 3

Let $S = [a, b]$. If

1. $(P_t)_{t \geqslant 0}$ is increasing
2. there exists a $u > 0$ such that P_u satisfies the MMC,

then $(P_t)_{t \geqslant 0}$ is globally stable with $\varphi^* = \varphi^* P$ and

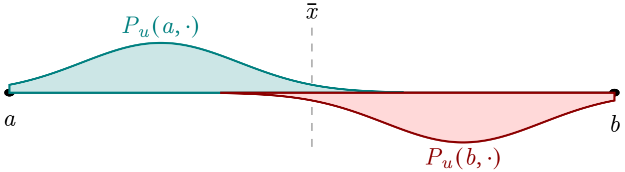
$$\beta(\varphi P_t, \varphi^*) \leqslant C e^{-\alpha t}$$

Remarks:

- Exponential convergence rate
- We also provide an ergodicity result

Proof Sketch

Under the MMC, everything can mix



Implies $X_t \preceq X'_t$ eventually, even if $X_0 = b$ and $X'_0 = a$

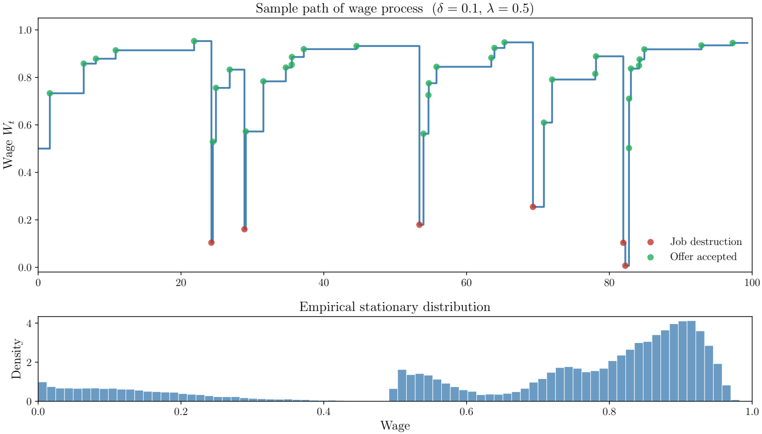
- Hence weak order mixing holds
- Tightness immediate because S compact

Example: Wage Dynamics

Continuous-time job ladder model on $S = [0, w_{\max}]$

- Job destruction at rate δ, new wage draw from $Q_u(w, \cdot)$
- Outside offers at rate λ: new wage offer from $Q_e(w, \cdot)$
  - keep max of current wage and offer
- Increasing: higher wages predict higher offers

Then Q_u, Q_e suitably mixing $\implies$ global stability


Sample path of wage process ($\delta = 0.1$, $\lambda = 0.5$)
Wage W_t
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
Job destruction
Offer accepted
Empirical stationary distribution
Density
4
2
0
0.0
0.2
0.4
0.6
0.8
1.0
Wage

Non-Compact State Example: Belief Shocks

Bayesian learning with occasional belief shocks

- Cogley and Sargent (2008), Orlik and Veldkamp (2014) etc.

Set up

- Agent faces uncertainty about hidden state $\theta \in \{\ell, h\}$
- Belief $\pi_t \in (0, 1)$ represents prob. that $\theta = h$ at time t
- Updated by Bayes' rule upon observing signal Z_{t+1}
- Occasionally reset by exogenous shocks

We work with log-odds

$$\eta_t := \ln\left(\frac{\pi_t}{1-\pi_t}\right)$$

Dynamics:

$$\eta_{t+1} = I_{t+1} \cdot R_{t+1} + (1 - I_{t+1}) \cdot (\eta_t + \xi_{t+1})$$

- $I_t \sim \text{Bernoulli}(\rho)$: reset indicator
- $R_{t+1} \sim Q(\eta_t, \cdot)$: reset value
- $\xi_{t+1} = \ln L(Z_{t+1})$: log-likelihood ratio from signal
- each Z_{t+1} drawn from f_h (the case f_ℓ case is similar)

If

1. (weak) positive correlation of beliefs at resets and
2. a moment condition on the log-likelihood ratio

then the belief process is globally stable

Weak order mixing proof: Run two copies (η_t) and (η'_t) from arbitrary initial conditions

- Same reset dates (shared indicator I_t)
- Independent reset draws, so $\mathbb{P}\{\eta_t \leqslant \eta'_t\} > 0$ at resets

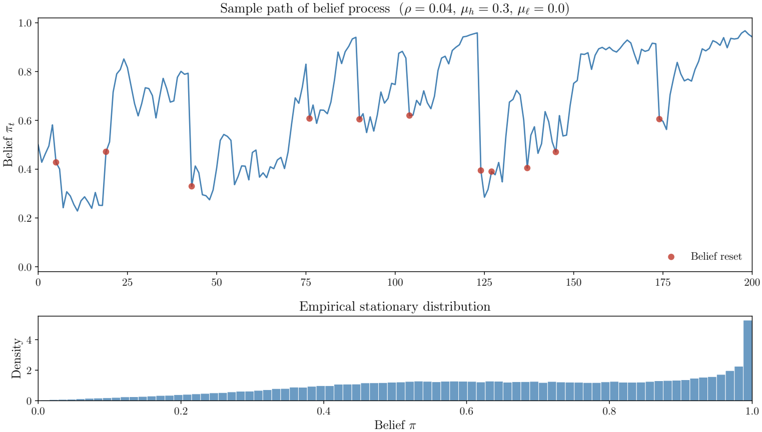


Figure: Belief dynamics: sample path and stationary distribution

Example: Income Dynamics Part I

Gabaix et al. (2016) study inequality dynamics

Log income follows a diffusion with jumps

$$\mathrm{d}X_t = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}B_t + \text{ resets}$$

Why Brownian motion?

- Ito's lemma
- Strong irreducibility properties

Alternative: **incomes constant between discrete events**

- promotions, layoffs, retirement, etc.

We model as a pure jump process $(X_t)_{t \geqslant 0}$ on $\mathbb{R}$

Two sources of shocks:

- Income shocks at rate λ_1

$$x \to x + \eta \qquad \text{with } \eta \sim F \text{ on } (0, \infty)$$

- Resets (death/replacement/etc) at rate λ_2

$$x \to h(x) + \zeta \qquad \text{with } \zeta \sim G \text{ on } \mathbb{R}$$

Proposition 1

If

1. h is increasing and bounded and
2. the support of G is sufficiently large

then the log income process is globally stable

Weak order mixing: Run two copies (X_t) and (X'_t) from arbitrary initial conditions

- Identical reset times and independent reset draws
- Independent resets induce ordering eventually

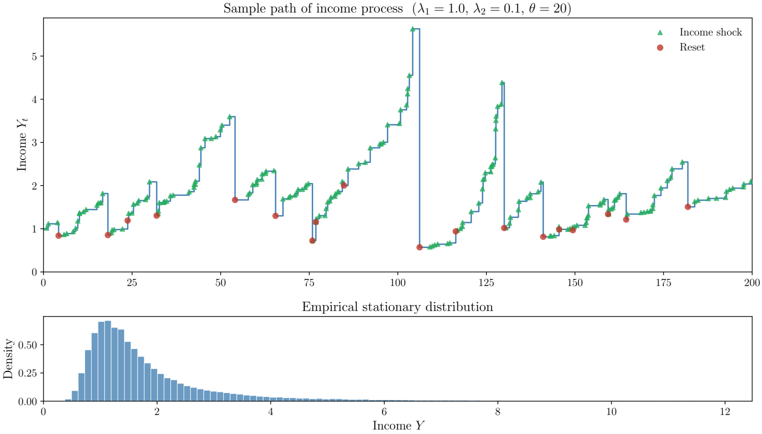


Figure: Pure jump income dynamics: sample path and stationary distribution

Example: Income Dynamics Part II

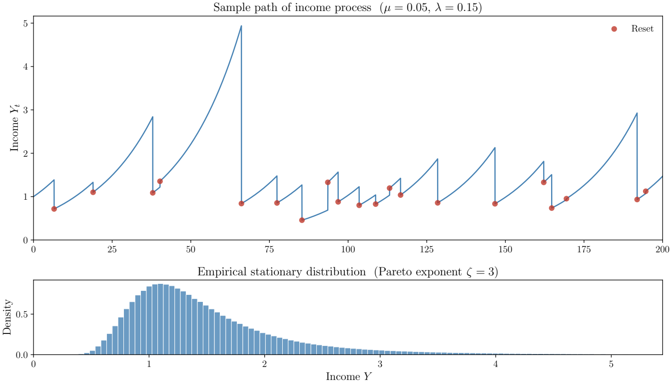
Previously we studied a pure jump process for income

Now suppose that $\dot{x} = g(x)$ between jumps

- A bit closer to Gabaix et al. (2016)
- Jumps occur at rate λ, only resets
- At jump times, $x \to h(x) + \zeta$ where $\zeta \sim G$

Proposition 2

If the conditions of Proposition 1 hold, then the income process is globally stable



Special case: constant drift $g(x) = \mu$ and deterministic resets to x_0 implies Pareto tail with exponent $\alpha := \lambda/\mu$

How Do The Proofs Work

Consider an order-preserving self-map T on a partially ordered metric space $E = (E, \preceq, d)$

Assumption 2

The space $(E, \preceq, d)$ is such that

1. d is complete
2. $d, \preceq$ satisfy the **diagonal property**:

$$d(f, g) \leqslant d(a, b) \quad \text{whenever} \quad f, g \in [a, b]$$

Holds for all Banach lattices ($\mathbb{R}^n$, function spaces, etc.)

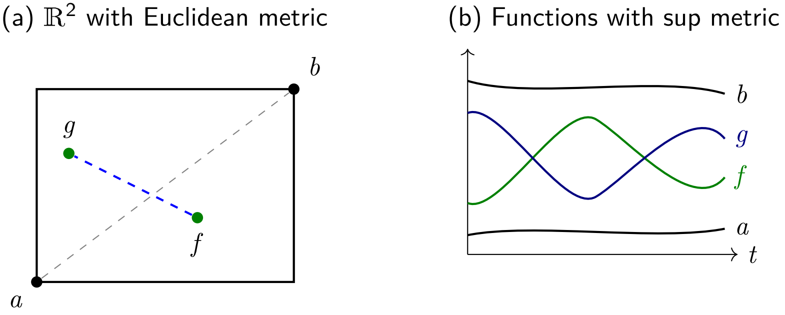


Figure: Diagonal property

Our main fixed point result:

Theorem 4: Discrete Time Fixed Point Theorem

If T is an order-preserving self-map on E, then

$$T \text{ is globally stable} \iff T \text{ is asymptotically contractive and has an order-bounded trajectory}$$

Proof: See the paper

We also extended this result to continuous time using semigroups

In the case $E = \mathscr{P}(S)$ we show that

- tightness = order bounded
- β is complete (Kamihigashi and Stachurski 2019)
- β has the diagonal property

Putting these facts together yields our main results:

1. asymptotically contractive + tight trajectory $\iff$ global stability
2. weakly order mixing + tight trajectory $\implies$ global stability

Summary

We show that, on $\mathscr{P}(S)$,

$$\textbf{global stability} \iff \textbf{asymp. contract.} + \textbf{tight trajectory}$$

We also show that

$$\text{weak order mixing} \implies \text{asymptotically contractive}$$

Benefits

- Weak conditions
- Handles continuous time and noncompact spaces

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