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Stochastic Optimal Growth with Unbounded Shock[1]

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This paper considers a neoclassical optimal growth problem where the shock that perturbs the economy in each time period is potentially unbounded on the state space. Sufficient conditions for existence, uniqueness, and stability of equilibria are derived in terms of the primitives of the model using recent techniques from the field of perturbed dynamical systems. *Journal of Economic Literature* Classification Numbers: C61, C62, O41.

Key Words: stochastic growth; Markov operators; Lagrange stability; strong contractiveness.

1. INTRODUCTION

This paper studies equilibria in the stochastic optimal growth economy of Brock and Mirman [5] without their assumption that the shock which perturbs production is realized within a bounded interval. It provides sufficient conditions for existence, uniqueness, and stability of equilibria in terms of the primitives of the one-sector model, namely the utility function u, the per capita production function f, and the distribution ψ of the disturbance term ε. The arguments are based on recent innovations in the theory of stochastically perturbed dynamical systems.

The original work of Brock and Mirman extends the deterministic optimal growth problem of Ramsey [34], Cass [7], Koopmans [22], and others to a stochastic setting. With regard to equilibria, they show that the existence, uniqueness, and stability results of the deterministic case are also realized in a stochastic model under similar assumptions on preferences and production technology. In their analysis, the productivity shock is restricted to a bounded interval of the real line.[2]

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[2] Such a shock is said to have compact support. For simplicity these shocks are referred to as "bounded." Shocks where no restrictions are placed on the support are called "unbounded."

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The problem of characterizing equilibria and long-run behavior in Brock–Mirman economies with bounded shock has subsequently been studied by Mirman [31], Mirman and Zilcha [32], Brock and Majumdar [6], Razin and Yahav [35], Donaldson and Mehra [8], Majumdar and Zilcha [29], Stokey *et al.* [37], Hopenhayn and Prescott [17], and Amir [2]. The analogous problem for the overlapping generations model with bounded shock has been studied by Laitner [24] and Wang [41]. The related question of ergodicity in moments for the Solow–Swan model with a shock that is unbounded above but cannot be arbitrarily small is investigated in Binder and Pesaran [4]. Evstigneev and Flåm [11] and Amir and Evstigneev [3] investigate the asymptotic distributions of aggregate rewards accumulated along equilibrium and optimal paths. More general studies of stochastic equilibria in economics include Futia [13] and Duffie *et al.* [9].

Stochastic growth with unbounded shock is treated in Mirman [30], who provides an existence result and proves that the equilibrium measure is not concentrated at zero. However, the sufficient conditions pertain to a class of consumption policies that may or may not be optimal (Mirman [30, A1–A3, p. 275]). In other words, the savings rate is exogenously given, and the conditions are not stated in terms of the primitives u, f, and ψ. Further, the problems of uniqueness and stability are not treated. In the present paper, conditions for existence, uniqueness, and stability are obtained in terms of the triple (u, f, ψ) and the restrictions imposed by optimizing behavior.

The mathematical techniques used in the paper are based on recent innovations in the theory of perturbed dynamical systems. The two key concepts are Lagrange stability and strong contractiveness. It is shown that an economy which is Lagrange stable has at least one equilibrium, and that an economy which is strongly contractive has at most one equilibrium. Moreover, for an economy with both properties, the unique equilibrium is globally asymptotically stable.[3]

In addition to identifying and characterizing equilibria in Brock–Mirman economies with unbounded shock, the paper also makes the following contributions. First, a formulation of Markovian systems known as the L_1 approach is introduced to stochastic growth theory. Second, the notions of strong contractiveness and Lagrange stability are developed in the context

[3] The primary mathematical reference is Lasota [25]. See also Lasota and Mackey [27] and Horbacz [19]. Previously, similar methods have been applied to the study of particle energy in an ideal gas (Lasota [25]), fluctuations in the brightness of the Milky Way (Lasota and Mackey [27]), propagation of annual plants with seed-bank (Horbacz [19]), and cell growth in a profilerating cell population (Lasota and Mackey [26], Tyson and Hannsgen [40], Tyrcha [39], Lasota *et al.* [28], and Lasota [25]).

of stochastic optimal growth. Third, new proofs are given for the two major fixed point results used in the paper. The first result states that every Lagrange stable Markov system has at least one fixed point, and the second that every strongly contractive and Lagrange stable Markov system is asymptotically stable. The proof of the former uses a Brouwer-type convexity argument, while that of the latter is based on the properties of contractive operators on a compact set.

Section 2 previews the mathematical arguments used in the paper. Section 3 formulates the stochastic optimal growth problem. Section 4 states the main result. The proof is then developed over Sections 5–7.

2. DISCUSSION OF TECHNIQUES

This section outlines the mathematical techniques used in the paper, with particular emphasis on the L_1 method for Markov processes.[4] The discussion is intended to be heuristic. Formal arguments are given following the statement of results in Section 4.

2.1. *Outline*

For dynamic economic models, an equilibrium (or steady state) is defined to be a point in the state space that is stationary under the period-to-period transition rule. If such a point is obtained, then no further change is observed in the system. As well as this invariance property, equilibria may be attractive for points in the surrounding state space, which is to say that the transition rule moves nearby points closer to the equilibrium.

In the case of stochastic models, a state cannot be stationary in the same sense as those in deterministic systems, given that shocks continue to disturb activity in each period. Instead, a steady state must be viewed as a situation where the probabilistic laws that govern the state variables cease to change over time (Green and Majumdar [14]). For stochastic economies the notion of stable equilibrium can be approached as follows. Since the path of the economy is a stochastic process, the state at any time in the future can be known only up to a probability distribution. Hence the state space is re-interpreted to be the collection of all density functions on the original space. Densities can be identified with points on the unit sphere in the space of integrable functions.[5] Thus any stable stochastic equilibrium

[4] Much of the early L_1 theory is due to Hopf [18]. The monograph of Foguel [12] contains an extensive survey of asymptotic results. Lasota and Mackey [27] use L_1 techniques to study perturbed and chaotic systems. Operator-theoretic treatment of Markov processes begins with Kryloff and Bogoliouboff [23]. See also Kakutani and Yoshida [21]. For an early operator-theoretic treatment of optimal stochastic growth see Brock and Majumdar [6].

[5] The set of densities coincides with the intersection of the positive cone and the boundary of the unit sphere.

can be viewed as a point on this infinite dimensional sphere to which nearby points are attracted as time evolves.

In this sense, deterministic and stochastic equilibria can be thought of as differing not conceptually but rather in the nature (in particular, in the dimension) of the space in which they are located. Here the above identification of stochastic equilibria with attractors on the unit sphere of the space of integrable functions is exploited to obtain sufficient conditions for the existence of stable equilibria in the stochastic neoclassical growth model.

2.2. *Discrete Dynamical Systems*

Consider in particular an abstract system characterized at each time t by a vector of state variables x_t taking values in state space U. Evolution is governed by a first-order difference equation

$$x_{t+1} = Tx_t, \qquad x \in U, \qquad T: U \to U. \tag{1}$$

The map T encodes the structure of the economic system, which is in turn determined by the primitives of the model, such as preferences, technology, and market conditions. A realization or *trajectory* for the system is a sequence $(T^n x)$ in U generated by iterating the map T on initial state x.[6] An equilibrium is a fixed point of T on U.

More generally, a *semidynamical system* is a pair (U, T), where U is a metric space and T is a continuous mapping of U into itself.[7] An *equilibrium* or *steady state* of (U, T) is a fixed point of T on U, i.e., a point $p \in U$ such that $Tp = p$. Fixed points are said to be *stationary* or *invariant* under T. Similar terminology also applies to sets. If $TA \subset A$, then A is said to be *invariant* under T. For fixed point p of T on U, the *stable set* $S_T(p)$ of p is that subset of U which is convergent to p under iteration of T:

$$S_T(p) = \{x \in U : T^n x \to p \ (n \to \infty)\}.$$

The point p is said to be *stable*, or an *attractor*, whenever there exists a set G open in U such that $p \in G$ and $S_T(p) \supset G$. In particular:

DEFINITION 2.1. Semidynamical system (U, T) is said to be *asymptotically stable* if there exists a unique fixed point p and $S_T(p) = U$.

[6] Here $T^n x = T(T^{n-1} x)$, $T^1 x = Tx$.

[7] The system is called *dynamical* if, in addition, the mapping T is invertible with continuous inverse (i.e., is a homeomorphism).

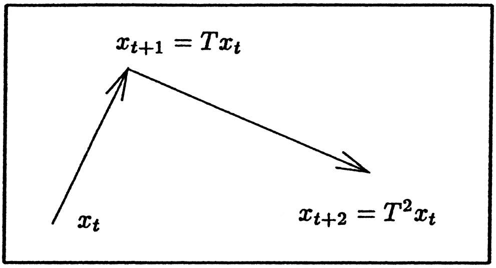

FIG. 1. Deterministic system in $\mathbb{R}^2$.

Figure 1 shows motion induced by iteration of an arbitrary map T, $U = \mathbb{R}^2$. Continued iteration generates a sequence in the plane.

2.3. *Stochastically Perturbed Dynamical Systems*

Suppose that the system (1) is perturbed at each transition from state x_t to state x_{t+1} by serially uncorrelated, U-valued shock ε_t with distribution given by density ψ:

$$x_{t+1} = T(x_t, \varepsilon_t), \qquad x \in U, \qquad \varepsilon_t \sim \psi. \tag{2}$$

For each fixed $x_t \in U$, x_{t+1} is a random variable with distribution uniquely determined by the value of x_t, the density ψ, and the map T. Let the density of this conditional distribution be $p(x_t, \cdot)$. That is,

$$p\colon U \times U \to \mathbb{R}, \qquad \text{Prob}(x_{t+1} \in B \mid x_t) = \int_B p(x_t, x_{t+1})\, dx_{t+1}, \tag{3}$$

where $\text{Prob}(x_{t+1} \in B \mid x_t)$ is the probability that the state vector is in $B \subset U$ at time $t+1$ given its current location at x_t. Figure 2 shows a perturbed system with additive shock in state space $\mathbb{R}^2$. The circles represent contour lines for the conditional density $p(x_t, \cdot)$. The bold arrows are sample realizations of the process.

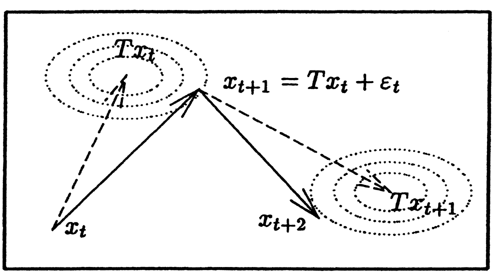

FIG. 2. The perturbed system.

The formulation (3) is convenient for calculation of the *unconditional* distribution of the state vector at each point in time. Suppose that the unconditional (marginal) distribution of x_t is known and is given by density φ_t. In this case,

$$\varphi_{t+1}(x_{t+1}) = \int p(x_t, x_{t+1})\, \varphi_t(x_t)\, dx_t \tag{4}$$

defines the unconditional density of the state at time $t+1$. The intuition is that the integral sums the probability $p(x_t, x_{t+1})$ of traveling to x_{t+1} from x_t for all $x_t \in U$, weighted at each point by the likelihood $\varphi_t(x_t)$ of x_t occurring as the current state. The recursion (4) provides a way to calculate the entire sequence of densities (φ_t) that represent the marginal distributions for the stochastic process (x_t) from any initial density φ_0 $(x_0 \sim \varphi_0)$.

In analyzing the behavior of the sequence (φ_t), one possibility is to use standard techniques from the classical theory of Markov processes (see, for example, Shiryaev [36, Chap. 8]). However, it is also possible to frame the same problem as a semidynamical system. The idea is to re-interpret the state space to be the collection of all densities on U. Call this set $D(U)$. The other half of the pair is the operator (call it P) that associates current-period with next-period densities through the integration defined in (4).

In this notation, (4) can be rewritten as

$$\varphi_{t+1} = P\varphi_t, \qquad \varphi \in D(U), \qquad P\colon D(U) \to D(U). \tag{5}$$

But the recursion (5) is now in exactly the same formula as the deterministic system (1), which means that similar techniques can be applied to its analysis. This translation of the perturbed system (2) into a deterministic map on the space of density functions is called the L_1 approach to Markov processes. Evolution of the economy is characterized by a sequence of densities generated by iterating P on some initial density φ_0. An equilibrium is a fixed point of the semidynamical system $(D(U)), P)$. The economy has a unique, globally stable equilibrium whenever $(D(U), P)$ is asymptotically stable in the sense of Definition 2.1.

These definitions are consistent with those used in previous studies.[8] However, the space of possible states $D(U)$ and hence equilibria has been constructed to include only those distributions that can be represented by density functions.[9] Thus probability mass cannot be concentrated at a

[8] The operator P is analogous to T^* in Brock and Majumdar [6, Eq. (4.3)], Futia [13, p. 380], and Stokey *et al.* [37, Eq. (2), p. 213], and to T in Hopenhayn and Prescott [17, p. 1392].

[9] The distributions which are absolutely continuous with respect to Lebesgue measure. For an earlier density-based treatment see Mirman [31].

point. In particular, this means that the model does not include the deterministic system as a special case; the distribution of the disturbance term ε must be non-degenerate.

3. FORMULATION OF THE PROBLEM

This section contains a formulation of the stochastic optimal growth problem studied by Brock and Mirman [5]. The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive reals, respectively. Given any metric space X, $\mathscr{B}(X)$ is the Borel sets of X. All sets of real numbers are Borel sets, and all real functions are Borel functions. Lebesgue measure is denoted by λ. Unless otherwise stated, integration is with respect to λ.

The accumulation problem evolves as follows. At the start of period t the (representative) agent receives income x_t. In response a level of consumption $c_t \leqslant x_t$ is chosen, yielding current utility $u(c_t)$. The remainder is invested in production, returning in the following period output $x_{t+1} = f(x_t - c_t)\, \varepsilon_t$. Here f is the production function and ε is a nonnegative random variable.[10] The process then repeats.

3.1. *Assumptions*

The functions u and f satisfy the usual assumptions.

Assumption 1. The production function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is zero at zero, strictly increasing, strictly concave, differentiable, and satisfies the Inada conditions $\lim_{x \downarrow 0} f'(x) = \infty$ and $\lim_{x \uparrow \infty} f'(x) = 0$.

Assumption 2. The utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable, and satisfies the interiority condition $\lim_{x \downarrow 0} u'(x) = \infty$.

The shock is permitted to be unbounded.

Assumption 3. The shocks to production are uncorrelated and identically distributed. The distribution of ε is represented by density ψ. The shock has finite mean $E(\varepsilon)$. In addition, ε satisfies $E(1/\varepsilon) < 1$. The shock is less than one with positive probability, i.e., $\int_0^1 \psi(x)\, dx \neq 0$.

[10] Following Stokey *et al.* [37] and Hopenhayn and Prescott [17], it is assumed that the disturbance term ε is multiplicative. Brock and Mirman use the more general formulation $x_{t+1} = f(x_t - c_t, \varepsilon_t)$. See Amir [2] for an even more general technology.

3.2. *Technology*

The conditional density for next-period output given income x and consumption c is, by a change of variable argument,

$$y \mapsto \psi\left(\frac{y}{f(x-c)}\right)\frac{1}{f(x-c)}. \tag{6}$$

Given that $f(0)=0$, (6) is not defined when consumption is equal to income. In this case (when $c=x$), next-period income is zero with probability one. Such a probability cannot be represented by a density. Consequently, the fully specified technology associating savings $x-c$ to next-period income will be defined by probability $B \mapsto \mathbf{Q}(x, c; B)$, where

$$\mathbf{Q}(x, c; B) = \int_B \psi\left(\frac{y}{f(x-c)}\right)\frac{1}{f(x-c)}\,dy,$$

when $c<x$, and by the probability concentrated at zero when $c=x$. Thus $\mathbf{Q}(x, c; B)$ is the probability that next-period output is in B given that current income is x and consumption is $c \in [0, x]$.

3.3. *The Optimal Policy*

Future utility is discounted geometrically at rate $\beta \in (0, 1)$. The agent selects a sequence (c_t) to solve

$$\max E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{7}$$

subject to the feasibility constraint $0 \leqslant c_{t+1} \leqslant f(x_t - c_t)\,\varepsilon_t$.

The meaning of the expectations operator in (7) is not immediately clear. A more formal statement of the problem is that the agent seeks a control policy $g\colon \mathbb{R}_+ \ni x_t \mapsto c_t \in \mathbb{R}_+$ that is feasible ($0 \leqslant g(x) \leqslant x$) and maximizes $v(x, g)$, where $v(x, g) = E_x^g[\sum_{t=0}^{\infty} \beta^t u(g(x_t))]$. Here E_x^g signals integration with respect to the (well-defined and unique) Markovian distribution over infinite-dimensional sequence space $\mathbb{R}_+^{\infty}$ generated by Markov transition kernel $\mathbf{Q}(x, g(x); dy)$.[11]

The value function V for the problem is defined at x as the supremum of $v(x, g)$ over the set of all feasible policies. A feasible policy g^* is called *optimal* if $v(x, g^*) = V(x)$ for all x.

The following results are well known.

[11] See, for example, Hernandez-Lerma and Lasserre [15].

THEOREM 3.1. *Let u, f, and ψ satisfy Assumptions 1–3. The following results hold.*

(1) *The value function V is finite and satisfies the Bellman equation*

$$V(x) = \max_{0 \leqslant c \leqslant x} \left\{ u(c) + \beta \int V(y)\, \mathbf{Q}(x, c; dy) \right\}. \tag{8}$$

(2) *There exists a unique optimal policy g and*

$$V(x) = u(g(x)) + \beta \int V(y)\, \mathbf{Q}(x, g(x); dy). \tag{9}$$

(3) *The value function is nondecreasing, concave, and differentiable with*

$$V'(x) = u'(g(x)). \tag{10}$$

(4) *If g is an optimal policy, then $0 < g(x) < x$, $\forall x > 0$, and both $x \mapsto g(x)$ and $x \mapsto x - g(x)$ are nondecreasing (savings and consumption both increase with income).*

Proof. See, for example, Mirman and Zilcha [32, pp. 331–332]. (For a formal discussion of Markov control programs with unbounded reward see Hernandez-Lerma and Lasserre [16, Chap. 8].) Here (1)–(3) $\Rightarrow$ (4). ∎

Substitution of the optimal control into the production relation yields the closed-loop law of motion

$$x_{t+1} = f(x_t - g(x_t))\, \varepsilon_t. \tag{11}$$

4. STATEMENT OF RESULTS

It is now possible to state the main result of the paper, which gives sufficient conditions for existence, uniqueness, and stability of equilibria in the stochastic growth model of the previous section. It shows that the results of Brock and Mirman also hold for many of the standard (unbounded) shocks used in mathematical statistics.

THEOREM 4.1. *Let u, f, and ψ satisfy Assumptions 1–3. The following statements are true.*

(1) *The economy* (u, f, ψ) *has at least one (nonzero) equilibrium.*

(2) *If, in addition,* ψ *is everywhere positive, then the equilibrium is unique and globally stable.*

The proof is developed in stages through the remaining sections. The approach is to represent the economy (u, f, ψ) as a semidynamical system and then apply two concepts used in the theory of such systems, namely Lagrange stability and strong contractiveness. In Section 5, Lagrange stability and strong contractiveness are defined. Further, it is shown (i) that every semidynamical system that is Lagrange stable and has certain linearity properties has at least one fixed point, and (ii) that every semidynamical system which is both Lagrange stable and strongly contractive is asymptotically stable. New proofs are offered for both results. In Section 6 it is shown that (u, f, ψ) can be represented as a semidynamical system. If it can be established under Assumptions 1–3 that this semidynamical system generated by (u, f, ψ) is Lagrange stable, then (i) can be used to demonstrate the existence of at least one equilibrium. If, in addition, it can be shown that positivity of ψ in part (2) of the theorem implies strong contractiveness, then by (ii) the system is also asymptotically stable, which is to say that there exists a unique and globally stable equilibrium. These two results are established in Section 7, completing the proof of the theorem.

The proof of Lagrange stability (Proposition 7.1) constitutes the main technical contribution of the paper. As expected, the Inada conditions and the concavity of the program are crucial to the proof.

4.1. *Examples*

Let f and u satisfy Assumptions 1 and 2, respectively, and let the density ψ of ε be lognormal. In other words, $\log \varepsilon$ is normally distributed with mean μ and variance σ^2, $\sigma > 0$. Since $E(1/\varepsilon) = \exp(\sigma^2/2 - \mu)$, $E(1/\varepsilon) < 1$ if $\sigma^2/2 < \mu$. In this case, all of the components of Assumption 3 are also satisfied, and (u, f, ψ) has at least one equilibrium. In addition, the density function is everywhere positive. It follows from part (2) of the theorem that the equilibrium is unique and globally stable.

In fact the same result holds for *any* lognormal shock. To see this, let μ and σ be arbitrary, $\sigma > 0$, and let θ be a constant strictly larger than $E(1/\varepsilon)$. If $\varepsilon^* = \theta\varepsilon$, $f^* = (1/\theta) f$, and ψ^* is the distribution of ε^*, then (u, f^*, ψ^*) satisfies Assumptions 1–3 and all of the conditions of the theorem. Hence (u, f^*, ψ^*) has a unique, globally stable equilibrium. But

$$f^*(x-c)\,\varepsilon^* = \frac{1}{\theta} f(x-c)\,\theta\varepsilon = f(x-c)\,\varepsilon,$$

so (u, f, ψ) and (u, f^*, ψ^*) are identical.[12] It follows that (u, f, ψ) also has a unique, globally stable equilibrium.

4.2. *Remarks*

The restriction

$$E(1/\varepsilon) = \int_0^\infty \frac{1}{x}\,\psi(x)\,dx < 1 \tag{12}$$

used in the theorem has a simple interpretation. Previous work has assumed that ε is realized in a compact interval $[a, b]$, $0 < a < b < \infty$. Here, in contrast, the shock may be arbitrarily large or arbitrarily close to zero. Equation (12) implies that ε is "unlikely" to be very close to zero, or, in other words, that the left-hand tail of the density ψ is relatively small. To see this, define for nonnegative summable functions V and h on $\mathbb{R}_{++}$ the (possibly infinite) number

$$E(V \mid h) = \int_{\mathbb{R}_{++}} V(x)\,h(x)\,dx, \tag{13}$$

as well as the set $G_a = \{x \in \mathbb{R}_{++} : V(x) < a\}$. Evidently,

$$E(V \mid h) \geqslant \int_{\mathbb{R}_{++}\setminus G_a} V(x)\,h(x)\,dx,$$

implying

$$\int_{\mathbb{R}_{++}\setminus G_a} h(x)\,dx \leqslant \frac{E(V \mid h)}{a}. \tag{14}$$

(This is a version of Chebychev's inequality.) Substituting $I^{-1}\colon x \mapsto x^{-1}$ for V, ψ for h, and $1/r$ for a gives

$$\int_0^r \psi(x)\,dx \leqslant rE(1/\varepsilon).$$

Thus (12) is a restriction on the left-hand tail of ψ.

It has also been assumed that $E(\varepsilon)$ is finite. This is a restriction on the right-hand tail. To see this, substitute $I\colon x \mapsto x$ for V and ψ for h to obtain

$$\int_a^\infty \psi(x)\,dx \leqslant \frac{E(\varepsilon)}{a}. \tag{15}$$

These restrictions on the tails of ψ can be thought of as a generalization of the assumption that ψ is zero below a and above b made in previous

[12] More formally, both economies have the same stochastic kernel. See Section 6.2.

studies. (As a caveat to the claim that the restrictions on ψ are a generalization of boundedness, recall that in this paper—in contrast to the majority of previous work—the shock must be non-degenerate and representable by a density function.)

The assumption on positivity of ψ in part (2) of the theorem is akin to the "communication" assumptions used in traditional Markov chain theory (Shiryaev [36, Chap. 8]).

5. LAGRANGE STABILITY AND CONTRACTIVE SYSTEMS

In this section the notions of Lagrange stability and strong contractiveness are developed and two fixed point results are established.

5.1. *Lagrange Stability*

Lagrange stability has been used extensively in the study of nonlinear differential equations and iterated function systems. Lagrange's original stability work was on the N-body problem of planetary motion. He showed that a first-order approximation of the system does not grow without bounds. The concept of Lagrange stability retains this meaning.

Recall that a set $A \subset U$ is called *precompact* if every sequence in A has a convergent subsequence. (A is compact if, in addition, the limit of the sequence is always in A.) A sequence (x_n) in U is defined to be precompact whenever $\{x_n: n \in \mathbb{N}\}$ is a precompact subset of U.

DEFINITION 5.1. Semidynamical system (U, T) is called *Lagrange stable* if the trajectory of x is precompact for every $x \in U$.[13]

A fixed point result for Lagrange stable systems is now stated. An alternative proof based on spectral decomposition can be found in Lasota and Mackey [27, Proposition 5.4.1], although the notation and formulation is somewhat different. Here a new proof is offered based on an infinite dimensional Brouwer fixed point theorem.

THEOREM 5.1. *Let X be a normed linear space, and let U be a nonempty convex closed subset of X. Let $T: X \to X$ be linear and continuous, with $TU \subset U$. If (U, T) is Lagrange stable, then T has a fixed point in U.*

Proof. Take any $x \in U$. Define $\gamma(x)$ to be the set $\{T^n x: n \in \mathbb{N}\}$, let $\hat{\gamma}(x)$ be its convex hull, and let $\mathrm{cl}(\hat{\gamma}(x))$ be the closure of the latter. Since the

[13] In finite dimensional space, precompactness is equivalent to boundedness by the Bolzano–Weierstrass theorem. Thus for such a space Lagrange stability corresponds to the idea that none of the possible trajectories for the state variables grow without bounds.

convex hull of a precompact set is again precompact, it follows that $\hat{\gamma}(x)$ is precompact. Since the closure of a precompact set is compact, $\mathrm{cl}(\hat{\gamma}(x))$ must be compact. Using the linearity of T, if $a \in \hat{\gamma}(x)$, then evidently Ta is again in $\hat{\gamma}(x)$, or $T\hat{\gamma}(x) \subset \hat{\gamma}(x)$. But then $T\,\mathrm{cl}(\hat{\gamma}(x)) \subset \mathrm{cl}(\hat{\gamma}(x))$.[14] Thus T is invariant on nonempty convex compact set $\mathrm{cl}(\hat{\gamma}(x))$. It follows that T has a fixed point in $\mathrm{cl}(\hat{\gamma}(x))$.[15] Finally, since $\mathrm{cl}(\hat{\gamma}(x)) \subset U$ by the assumption that U is closed and convex, the fixed point must also be in U. ∎

5.2. *Strongly Contractive Systems*

Next strong contractiveness and its relationship to Lagrange stability is discussed.

In many fields of economics, Banach's contraction principle is used to locate equilibria and solve dynamic programs.[16] Let U be a metric space and let $T: U \to U$. T is said to be a contraction mapping in the sense of Banach if there exists an $\alpha < 1$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y), \qquad \forall x, y \in U. \tag{16}$$

Banach's contraction principle is equivalent to the statement that if U is complete and T satisfies (16), then semidynamical system (U, T) is asymptotically stable (Joshi and Bose [20, Theorem 4.1.1]).

Unfortunately, for the semidynamical systems generated by stochastic growth models with unbounded shock, (16) either does not hold or is difficult to verify. In contrast, the slightly weaker condition (18) below will be shown to be an immediate consequence of positivity of the distribution ψ. (Recall Theorem 4.1, part (2).)

DEFINITION 5.2. Semidynamical system (U, T) is called *contractive* if

$$d(Tx, Ty) \leqslant d(x, y), \qquad \forall x, y \in U. \tag{17}$$

The system is called *strongly contractive* if, in addition,

$$d(Tx, Ty) < d(x, y), \qquad \forall x, y \in U, \quad x \neq y. \tag{18}$$

[14] If A is any set with $TA \subset A$ and A' is the closure of A, then $a' \in A'$ implies the existence of a sequence $(a_n) \subset A$, $a_n \to a'$. But then $Ta' = T \lim a_n = \lim Ta_n$, which, as the limit of a sequence in A, must again be in A'. Hence $TA' \subset A'$.

[15] This is the Brouwer argument. See Joshi and Bose [20, Theorem 4.3.10].

[16] See, for example, Stokey *et al.* [37, Lemma 11.11 and Sect. 17.2].

Evidently (16) $\Rightarrow$ (18) $\Rightarrow$ (17). Like contractiveness in the sense of Banach, strong contractiveness implies uniqueness of equilibrium. (Suppose otherwise. In particular, let distinct points x and y be stationary under T. Then both $d(x, y) = d(Tx, Ty)$ and $d(Tx, Ty) < d(x, y)$, a contradiction.) However, strong contractiveness does not guarantee existence. (For example, consider $U = \mathbb{R}_+$, $T: x \mapsto x + e^{-x}$.) Nevertheless, existence and stability can be obtained if strong contractiveness is supplemented by compactness of U:

LEMMA 5.1. *Let (U, T) be a semidynamical system. If (U, T) is strongly contractive and U is compact, then (U, T) is asymptotically stable.*

Proof. See Joshi and Bose [20, Theorem 4.1.6, Corollary 1]. ∎

In the arguments that follow, the underlying space U corresponds to the space of density functions on $\mathbb{R}_{++}$, which is defined below as the intersection of the positive cone and the boundary of the unit sphere in the space of summable functions. This set is not compact, and hence Lemma 5.1 is not immediately applicable. However, it is closed, and in this case compactness can be replaced by Lagrange stability. For discrete dynamical systems this fact was recently proved in the context of Hausdorff space using Liapunov methods (Lasota [25, Theorem 2.1]). Here a simple new proof is given.

THEOREM 5.2. *Let X be a metric space, let U be a nonempty closed subset of X, and let $T: X \to X$ be a continuous function invariant on U. If (U, T) is both Lagrange stable and strongly contractive, then it is asymptotically stable.*

Proof. Fix $x \in U$. Define $\Gamma(x)$ to be the closure of $\{T^n x: n \in \mathbb{N}\}$. Since (U, T) is Lagrange stable, $\Gamma(x)$ is a compact subset of X. Moreover, $T\Gamma(x) \subset \Gamma(x)$. Therefore $(\Gamma(x), T)$ is itself a strongly contractive semidynamical system on a compact set, and, by Lemma 5.1, has a unique fixed point $p \in \Gamma(x)$ with $T^n x \to p$. The point p is in U because U is closed and hence $\Gamma(x) \subset U$. Moreover, (U, T) has at most one fixed point by strong contractiveness. Therefore p does not depend on x. The result follows. ∎

6. MARKOV CHAINS AS SEMIDYNAMICAL SYSTEMS

In this section it is shown that (u, f, ψ) can be interpreted as a semidynamical system. Mathematically, the exposition is based on Hopf [18], Foguel [12], Lasota [25], and Lasota and Mackey [27]. Although the formal structure is somewhat different, our approach to Markovian growth models benefits from previous operator-theoretic treatments, such as Brock and Majumdar [6], Futia [13], Stokey *et al.* [37], and Hopenhayn and Prescott [17].

Consider the perturbed system (2). Let Σ be a σ-algebra of subsets of U, and let ν be a σ-finite measure on (U, Σ). As usual, $L_1(U)$ denotes the Banach lattice of ν-integrable functions on U with norm $\|f\| = \int |f| \, d\nu$ and pointwise ordering. Functions in $L_1(U)$ are defined only up to the complement of a ν-null set, and "almost everywhere" notation is suppressed throughout. The symbol $\Sigma \otimes \Sigma$ denotes the σ-algebra on $U \times U$ generated by sets in $\Sigma \times \Sigma$.

A formal representation of p in (3) is now possible:

DEFINITION 6.1. A *stochastic kernel* for measure space (U, Σ, ν) is a nonnegative, real-valued, and $\Sigma \otimes \Sigma$-measurable function p on $U \times U$ such that

$$\int p(x, y) \, \nu(dy) = 1, \qquad \forall x \in U.$$

Thus $\{y \mapsto p(x, y)\} \in L_1(U)$ is a density for each $x \in U$.

6.1. *Markov Operators*

Let X be an ordered normed space. That is, X is a normed vector space with partial order $\leqslant$ such that $x \leqslant y$ implies $x + z \leqslant y + z$, $\forall z \in X$, and $\alpha x \leqslant \alpha y$ for nonnegative scalar α. The order $\leqslant$ defines a *positive cone* $X_+ = \{x \in X : 0 \leqslant x\}$. An operator P mapping X into itself is called *positive* if $PX_+ \subset X_+$, and *isometric* on $A \subset X$ if $\|Pa\| = \|a\|$ whenever $a \in A$.

The following definition is a generalization of Hopf [18, Definition 2.1].

DEFINITION 6.2. Let X be an ordered normed space. A *Markov operator* on X is a linear operator $P\colon X \to X$ such that P is (i) positive and (ii) isometric on the positive cone X_+.[17]

To each stochastic kernel p on (U, Σ, ν) corresponds a linear operator $P\colon L_1(U) \to L_1(U)$ defined by

$$(Pf)(y) = \int p(x, y) \, f(x) \, \nu(dx). \tag{19}$$

[17] Markov operators in our sense are often called *stochastic operators* in the literature on positive operators on AL and AM spaces.

The operator P is a Markov operator on $L_1(U)$.[18] Here (19) is the operation in (4), and P corresponds to the operator in (5).

Since P is a positive linear operator from a Banach lattice into itself, it follows that P is continuous.[19] In fact P is a contraction.

LEMMA 6.1 (Lasota and Mackey). *If P is a Markov operator on $L_1(U)$, then $(L_1(U), P)$ is contractive.*

Proof. Fix $f \in L_1(U)$. Define $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. By linearity and positivity,

$$|Pf(x)| = |Pf^+(x) - Pf^-(x)| \leqslant Pf^+(x) + Pf^-(x) = P\,|f(x)|.$$

Integration obtains

$$\|Pf\| = \int |Pf|\, dv \leqslant \int P\,|f|\, dv = \|f\|.$$

An application of linearity yields (17). ∎

Let P be a Markov operator on $L_1(U)$, and let $D(U)$ be the collection of all densities on $L_1(U)$. That is,

$$D(U) = \{f \in L_1(U) : f \geqslant 0 \text{ and } \|f\| = 1\}.$$

The space $D(U)$ is a metric space in the norm distance inherited from $L_1(U)$. It is clear from Definition 6.2 that $PD(U) \subset D(U)$.[20] Since P is continuous, $(D(U), P)$ is a semidynamical system.

6.2. *The Brock–Mirman Process*

The stochastic kernel, Markov operator, and semidynamical system associated with the Brock–Mirman economy (u, f, ψ) are derived from the law of motion (11).

By a change of variable argument, the conditional density for next-period income given that current income equals x is

$$y \mapsto \psi\left(\frac{y}{f(x-g(x))}\right)\frac{1}{f(x-g(x))}. \tag{20}$$

[18] Evidently P is positive. That $\|Pf\| = \|f\|$ when $f \geqslant 0$ follows from an application of Fubini's theorem.

[19] See, for example, Aliprantis and Burkinshaw [1, Theorem 12.3].

[20] This property characterizes Markov operators on $L_1(U)$.

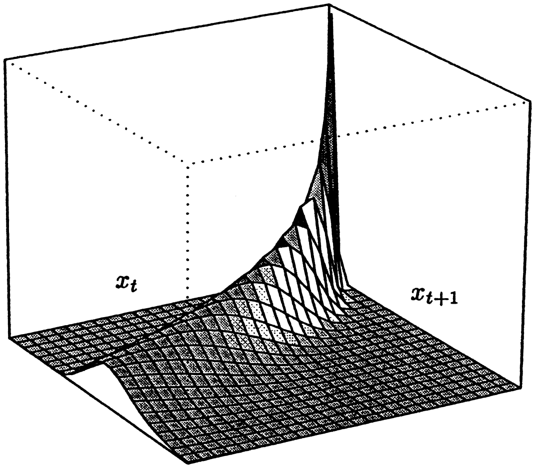

FIG. 3. Stochastic kernel (20).

As a function of both x and y, (20) defines a stochastic kernel for measure space $(\mathbb{R}_{++}, \mathscr{B}(\mathbb{R}_{++}), \lambda)$ in the sense of Definition 6.1. Denote by Q the Markov operator associated with (20) by (19). The semidynamical system for the Brock–Mirman process is then $(D(\mathbb{R}_{++}), Q)$, where $D(\mathbb{R}_{++})$ is the densities on $\mathbb{R}_{++}$. If initial income x_0 is distributed according to φ_0, then time t income is distributed according to $Q^t \varphi_0$.

A plot of (20) is shown in Fig. 3 for the parameterization $f: x \mapsto x^{1/2}$, $u: x \mapsto \log x$, ε lognormal. The origin is the corner of the graph furthest from the viewer. For each x_t, a density function runs parallel to the x_{t+1} axis. The density governs the likelihood that income per head takes values along that axis, given that the current state is x_t.[21]

7. PROOF OF THE MAIN THEOREM

The proof of Theorem 4.1 proceeds as follows. In Section 7.1 Lagrange stability of (u, f, ψ) is established. In Section 7.2 strong contractiveness of the economy is established using the additional hypothesis of positivity of ψ. The proof is completed in Section 7.3.

7.1. *Proof of Lagrange Stability*

The first lemma provides a way to identify weakly precompact sets in $L_1(\mathbb{R}_{++})$.

[21] For a kernel estimated nonparametrically from actual growth data see Quah [33, Figs. 5 and 6].

LEMMA 7.1. *Let $\mathscr{M}$ be a bounded set of nonnegative functions in $L_1(\mathbb{R}_{++})$. $\mathscr{M}$ is weakly precompact if the following two conditions hold.*

(1) *For all $\varepsilon > 0$, $\exists \delta > 0$ such that if $A \in \mathscr{B}(\mathbb{R}_{++})$ and $\lambda(A) < \delta$, then*

$$\int_A h(x)\,dx < \varepsilon, \qquad \forall h \in \mathscr{M}.$$

(2) *There exists a constant M such that $\forall r > 0$,*

$$\int_r^\infty h(x)\,dx \leqslant \frac{M}{r}, \qquad \forall h \in \mathscr{M}.$$

Proof. See Dunford and Schwartz [10, IV.13.54]. ∎

The next lemma is required for the proof of Lagrange stability of (u, f, ψ).

LEMMA 7.2. *Let (u, f, ψ) satisfy Assumptions 1–3. If g is an optimal policy, then there exists an $x_0 > 0$ such that*

$$f(x - g(x)) \geqslant x \qquad \textit{whenever} \quad x \in (0, x_0].$$

Proof. The first order condition of (8) is

$$u'(g(x)) = \beta \int_0^\infty V'(f(x-g(x))\,z)\, f'(x-g(x))\, z\psi(z)\,dz.$$

Using the envelope relation (10) obtains

$$\begin{aligned} V'(x) &= \beta \int_0^\infty V'(f(x-g(x))\,z)\, f'(x-g(x))\, z\psi(z)\,dz \\ &\geqslant \beta \int_0^1 V'(f(x-g(x))\,z)\, f'(x-g(x))\, z\psi(z)\,dz \\ &\geqslant \beta \int_0^1 V'(f(x-g(x)))\, f'(x-g(x))\, z\psi(z)\,dz, \end{aligned}$$

where the first inequality follows from the fact that V is nondecreasing and the second from the fact that V is concave. Thus,

$$V'(x) \geqslant V'(f(x-g(x)))\, f'(x-g(x))\, M, \quad M = \beta \int_0^1 z\psi(z)\, dz.$$

The constant M is positive by Assumption 3. Assumption 1 and the monotonicity of $x \mapsto x-g(x)$ then imply the existence of an $x_0 > 0$ such that $f'(x-g(x))\, M \geqslant 1$ whenever $x \in (0, x_0]$. Therefore,

$$V'(x) \geqslant V'(f(x-g(x))) \qquad \text{on} \quad (0, x_0].$$

The result now follows from the concavity of V. ∎

The proof of the following proposition, which is the central technical result in the paper, draws heavily on methods developed by Lasota [25] and Horbacz [19].

PROPOSITION 7.1. *If* (u, f, ψ) *satisfies Assumptions* 1–3, *then the associated semidynamical system* $(D(\mathbb{R}_{++}), Q)$ *is Lagrange stable.*

Proof. It follows from Lasota [25, Proposition 3.4 and Theorem 4.1] that to establish precompactness of $(Q^n\varphi)$ for any density φ it is sufficient to find a set $\mathscr{M} \subset D(\mathbb{R}_{++})$ such that $\mathscr{M}$ is dense in $D(\mathbb{R}_{++})$ and $(Q^n h)$ is weakly precompact for every h in $\mathscr{M}$.[22]

Let $\mathscr{M}$ be the collection of densities h that satisfy

$$\int_0^\infty xh(x)\, dx < \infty \qquad \text{and} \qquad \int_0^\infty \frac{1}{x}\, h(x)\, dx < \infty. \tag{21}$$

We claim that $\mathscr{M}$ has the desired properties. To see that $\mathscr{M}$ is dense in the densities, fix $\varphi \in D(\mathbb{R}_{++})$ and define $h_k^0 = \mathbf{1}_{(1/k,\, k)}\varphi$.[23] Since $\|h_k^0\| \uparrow 1$ by the monotone convergence theorem, it follows that for some $K \in \mathbb{N}$, $\|h_k^0\| > 0$ whenever $k \geqslant K$. For all such k define

$$h_k = \|h_k^0\|^{-1}\, h_k^0.$$

It can be established that h_k satisfies (21) for each k. In addition, h_k is a density by construction, and $h_k \to \varphi$ pointwise. But then $h_k \to \varphi$ in the L_1 norm by Scheffe's lemma.[24] Thus $\mathscr{M}$ is dense in $D(\mathbb{R}_{++})$.

It remains to show that if $h \in \mathscr{M}$ then $(Q^n h)_{n \geqslant 1}$ is weakly precompact. Fix arbitrary $h \in \mathscr{M}$. It is sufficient to establish precompactness of $(Q^n h)_{n \geqslant N}$ for some fixed $N \in \mathbb{N}$, because appending a finite number of elements to a (weakly) precompact set does not alter the property of (weak)

[22] This holds for any integral Markov operator, that is, any Markov operator derived from a stochastic kernel via (19). See Lasota [25].

[23] Here $\mathbf{1}_{(1/k,\, k)}$ is the characteristic function of $(1/k, k)$.

[24] See, for example, Taylor [38, Proposition 4.5.14].

precompactness. We now show that $(Q^n h)_{n \geqslant N}$ is weakly precompact by verifying the conditions of Lemma 7.1.

Boundedness of the collection is satisfied because $\|Q^n h\| = \|h\| = 1$ for all n by the positive isometry property of Markov operators (Definition 6.2). We now show that conditions (1) and (2) also hold.

For notational simplicity define $q(x) = f(x - g(x))$. Use will be made of the fact that

$$\frac{1}{q(x)} E(1/\varepsilon) \leqslant \gamma \frac{1}{x} + C \tag{22}$$

for all positive x, where γ and C are nonnegative constants, $\gamma < 1$.

To verify (22), recall that $\exists x_0 > 0$ such that $q(x) \geqslant x$ when $x \leqslant x_0$ by Lemma 7.2. Choose any γ such that $E(1/\varepsilon) < \gamma < 1$. Then

$$\frac{1}{q(x)} E(1/\varepsilon) \leqslant \gamma \frac{1}{x}, \qquad \forall x \leqslant x_0. \tag{23}$$

Moreover, on $[x_0, \infty)$, monotonicity of f and $x \mapsto x - g(x)$ implies that $q(x) \geqslant q(x_0)$, or

$$\frac{1}{q(x)} E(1/\varepsilon) \leqslant \frac{1}{q(x_0)} E(1/\varepsilon) = C. \tag{24}$$

Together, (23) and (24) imply (22).

Let I^{-1} be the map $x \mapsto x^{-1}$. Applying in succession (13), (19), Fubini's theorem, a change of variable argument, and (22),

$$\begin{aligned}
E(I^{-1} \mid Q^n h) &= \int_0^\infty \frac{1}{y} Q^n h(y)\, dy \\
&= \int_0^\infty \frac{1}{y} \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)} Q^{n-1} h(x)\, dx \right] dy \\
&= \int_0^\infty \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)} \frac{1}{y}\, dy \right] Q^{n-1} h(x)\, dx \\
&= \int_0^\infty \frac{1}{q(x)} E(1/\varepsilon)\, Q^{n-1} h(x)\, dx \\
&\leqslant \int_0^\infty \left[\gamma \frac{1}{x} + C \right] Q^{n-1} h(x)\, dx \\
&= \gamma E(I^{-1} \mid Q^{n-1}) + C.
\end{aligned}$$

Repeating the argument n times,

$$E(I^{-1} \mid Q^n h) \leqslant \gamma^n E(I^{-1} \mid h) + \frac{C}{1-\gamma},$$

or, using finiteness of $E(I^{-1} \mid h)$,

$$E(I^{-1} \mid Q^n h) \leqslant 1 + \frac{C}{1-\gamma}$$

when $n \geqslant K$, K suitably large.

An application of the Chebychev argument (14) gives

$$\int_0^r Q^n h(x)\, dx \leqslant r E(I^{-1} \mid Q^n h)$$

for any positive r. Therefore,

$$\int_0^r Q^n h(x)\, dx \leqslant r\left(1 + \frac{C}{1-\gamma}\right), \qquad n \geqslant K. \tag{25}$$

Now fix any $\varepsilon > 0$. According to Lemma 7.1 part (1), we require a $\delta > 0$ and a $K \in \mathbb{N}$ such that $n \geqslant K$ implies

$$\int_A Q^n h(x)\, dx < \varepsilon$$

whenever $\lambda(A) \leqslant \delta$. For this purpose, consider the decomposition

$$\int_A Q^n h(x)\, dx = \int_{A \cap (0,r)} Q^n h(x)\, dx + \int_{A \cap (r,\infty)} Q^n h(x)\, dx. \tag{26}$$

Using (25) gives

$$\int_{A \cap (0,r)} Q^n h(x)\, dx \leqslant \int_0^r Q^n h(x)\, dx < \frac{\varepsilon}{2}, \tag{27}$$

when $r > 0$ is chosen to be sufficiently small and $n \geqslant K$.

Take r as given and consider the second term in (26),

$$\begin{aligned}
\int_{A \cap (r,\infty)} Q^n h(x)\, dx &= \int_{A \cap (r,\infty)} \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)}\, Q^{n-1} h(x)\, dx\right] dy \\
&= \int_0^\infty \left[\int_{A \cap (r,\infty)} \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)}\, dy\right] Q^{n-1} h(x)\, dx \\
&= \int_0^\infty \left[\int_{\frac{A \cap (r,\infty)}{q(x)}} \psi(z)\, dz\right] Q^{n-1} h(x)\, dx.
\end{aligned}$$

The term in brackets can be written as

$$G(x) = \int_{\frac{r}{q(x)}}^{\infty} \mathbf{1}_{\frac{A}{q(x)}}(z)\, \psi(z)\, dz.$$

By (15) it is possible to choose an $\alpha > 0$ so small that

$$\int_{\frac{r}{q(\alpha)}}^{\infty} \psi(z)\, dz < \frac{\varepsilon}{2}.$$

Evidently,

$$G(x) < \frac{\varepsilon}{2} \qquad \text{whenever} \quad x \leqslant \alpha.$$

Now consider the case where $x > \alpha$. Select $\delta' > 0$ such that

$$\lambda(B) < \delta' \Rightarrow \int_B \psi(z)\, dz < \frac{\varepsilon}{2}.$$

(Existence of such a δ' follows from absolute continuity of the integral measure with respect to λ. See, for example, Taylor [38, 2.8.15].) Define $\delta = q(\alpha)\, \delta'$. Then $x > \alpha$ and $\lambda(A) < \delta$ implies

$$G(x) \leqslant \int_{\frac{A}{q(x)}} \psi(z)\, dz < \frac{\varepsilon}{2},$$

because $\lambda(A/q(x)) = \lambda(A)/q(x) \leqslant \lambda(A)/q(\alpha) < \delta/q(\alpha) = \delta'$. Thus $\lambda(A) < \delta$ implies $G(x) < \varepsilon/2$ for all x, and hence

$$\int_{A \cap (r, \infty)} Q^n h(x)\, dx < \frac{\varepsilon}{2}. \tag{28}$$

Combining (26), (27), and (28) gives

$$\int_A Q^n h(x)\, dx < \varepsilon$$

when $\lambda(A) < \delta$ and $n \geqslant K$. Thus condition (1) of Lemma 7.1 holds for the collection $(Q^n h)_{n \geqslant K}$.

Next, condition (2) of the lemma needs to be checked for the same h. Let I be the identity map on $\mathbb{R}_{++}$. We have

$$\begin{aligned}
E(I \mid Q^n h) &= \int_0^\infty y Q^n h(y)\, dy \\
&= \int_0^\infty y \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)}\, Q^{n-1} h(x)\, dx \right] dy \\
&= \int_0^\infty \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)}\, y\, dy \right] Q^{n-1} h(x)\, dx \\
&= \int_0^\infty E(\varepsilon)\, q(x)\, Q^{n-1} h(x)\, dx \\
&\leqslant \int_0^\infty E(\varepsilon)\, f(x)\, Q^{n-1} h(x)\, dx.
\end{aligned}$$

Since $E(\varepsilon)$ is finite, it follows from the concavity and Inada conditions in Assumption 1 that $x \mapsto E(\varepsilon)\, f(x)$ can be majorized on $\mathbb{R}_{++}$ by an affine function with slope less than one. In other words, there exist nonnegative constants a and b, $a < 1$, such that $E(\varepsilon)\, f(x) \leqslant ax + b$, $\forall x > 0$. Therefore,

$$\begin{aligned}
\int_0^\infty E(\varepsilon)\, f(x)\, Q^{n-1} h(x)\, dx &\leqslant \int_0^\infty [ax + b]\, Q^{n-1} h(x)\, dx \\
&= aE(I \mid Q^{n-1} h) + b.
\end{aligned}$$

Repeating the argument n times,

$$E(I \mid Q^n h) \leqslant a^n E(I \mid h) + \frac{b}{1-a},$$

or, using finiteness of $E(I \mid h)$,

$$E(I \mid Q^n h) \leqslant 1 + \frac{b}{1-a}$$

when $n \geqslant M$, M suitably large.

By (14),

$$\int_r^\infty Q^n h(x)\, dx \leqslant \frac{E(I \mid Q^n h)}{r}$$

for any n and any positive r. Hence

$$\int_r^\infty Q^n h(x)\, dx \leqslant \frac{1}{r}\left(1 + \frac{b}{1-a}\right), \qquad n \geqslant M,$$

and condition (2) of Lemma 7.1 holds for $(Q^n h)_{n \geqslant M}$.

Finally, define $N = \max(K, M)$. Then $(Q^n h)_{n \geqslant N}$ satisfies all of the conditions of the lemma, completing the proof of Lagrange stability. ∎

7.2. *Proof of Strong Contractiveness*

A well-known sufficient condition for strong contractiveness of Markov operators on $L_1(U)$ is as follows.

LEMMA 7.3. *For given measure space (U, Σ, v), let p be a stochastic kernel, and let P be the associated Markov operator. If $p > 0$ on $U \times U$, then $(D(U), P)$ is strongly contractive.*

Proof. See Lasota [25, Proposition 3.1]. ∎

PROPOSITION 7.2. *If ψ is everywhere positive, then $(D(\mathbb{R}_{++}), Q)$ is strongly contractive.*

Proof. Immediate from (20) and Lemma 7.3. ∎

7.3. *Proof of Theorem* 4.1

It is now possible to prove Theorem 4.1. Proposition 7.1 shows that if (u, f, ψ) satisfies Assumptions 1–3, then the associated semidynamical system $(D(\mathbb{R}_{++}), Q)$ is Lagrange stable. Evidently $D(\mathbb{R}_{++})$ is a closed convex subset of $L_1(\mathbb{R}_{++})$. Moreover Q is both linear and continuous. Hence all the conditions of Theorem 5.1 are satisfied, implying the existence of an equilibrium density. Since the equilibrium is a density, probability is not concentrated at zero (i.e., it is a nonzero equilibrium). Regarding part (2) of the theorem, if, in addition, ψ is assumed to be everywhere positive, then $(D(\mathbb{R}_{++}), Q)$ is also strongly contractive by Proposition 7.2. Existence, uniqueness, and stability of equilibrium now follow from Theorem 5.2.

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