Stability of Stochastic Optimal Growth Models: A New Approach ⋆

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Abstract

The paper proposes an Euler equation technique for analyzing the stability of differentiable stochastic programs. The main innovation is to use marginal reward directly as a Foster–Lyapunov function. This allows us to extend known stability results for stochastic optimal growth models, both weakening hypotheses and strengthening conclusions. *JEL classification:* C61; C62; O41

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1 Introduction

Many economic models are now explicitly dynamic and stochastic. Their state variables evolve in line with the decisions and actions of individual economic agents. These decisions are identified in turn by imposing rationality. Depending on technology, market structure, time discount rates and other primitives, rational behavior may lead either to stability or to instability. [1]

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[1] Rationality itself has no intrinsic stability implications—any sufficiently smooth function can be rationalized as the solution to a discounted dynamic program [4].

A classic study of stability in dynamic stochastic models is Brock and Mirman [5]. They show that for many convex one-sector growth models, the optimizing behavior of agents implies convergence for the sequence of per-capita income distributions to a unique, nondegenerate limiting distribution, or stochastic steady state. Put differently, the optimal process for the state variable is ergodic.

Their study laid the foundations for a vast and growing literature, spanning economic development, public finance, fiscal policy, environmental and resource economics, monetary policy and asset pricing. For much of this research the existence of a unique nontrivial long-run equilibrium distribution is fundamental (implicitly, often explicitly).

In this paper we illustrate a new method for determining whether optimal accumulation problems are stable. It combines the Euler equation of the optimal program with the Foster–Lyapunov theory of Markov chains. In fact the Euler equation is identified directly with a Lyapunov equation, connecting stochastic optimal growth with strong ergodicity results from Foster–Lyapunov theory.

The inherent simplicity of this technique allows us to eliminate several conditions required for ergodicity in earlier studies. For example, technology need not be convex, productivity shocks need not be bounded, and marginal product of capital at zero need not be infinite: We give a precise condition on the latter in terms of the shock distribution and the agent's discount factor.

The problem of establishing a general stability result for *nonconvex* stochastic optimal growth has resisted fundamental innovation for many years. This is not because convexity is somehow intrinsic to stable investment behavior. Rather it is due to the difficulty of untangling the implications of the Euler equation without the extra structure that convexity provides. For it appeared that only when these implications had been recovered could work begin on finding appropriate Lyapunov functions (or applying some other standard dynamics machinery). In this paper the Euler equation *is* the Lyapunov equation. We find that under standard productivity shock distributions local nonconvexities do not alter Brock and Mirman's essential conclusion.

Research on Lyapunov techniques for studying Markov dynamics is still very active. For a recent survey see the (excellent) monograph of Meyn and Tweedie [15]. We exploit in particular the powerful theory of V-uniform ergodicity [15, Ch. 16]. By linking our Euler equation method with this theory, we prove that in general optimal stochastic growth models are not only ergodic but *geometrically* ergodic. That is, for any given starting point, the distance between the current distribution and the limiting distribution decreases at a geometric rate.

Geometric ergodicity has numerous theoretical and empirical applications. As an example of the former, the *rate* at which stochastically growing economies tend to their steady state is a central component of the "convergence" debate; of the latter, geometric convergence is required by Duffie and Singleton [8] for consistency of the Simulated Moments Estimator. Geometric ergodicity also has applications to numerical procedures: When computing ergodic distributions, rates of convergence can determine bounds on algorithm run-times for a prescribed level of accuracy.

Finally, we use V-uniform ergodicity to prove that under standard econometric assumptions on the noise process the series for the state variable also satisfies both the Law of Large Numbers (LLN) and Central Limit Theorem (CLT). The former states that sample means of the optimal process converge asymptotically to their long-run expected value. The latter associates asymptotic distributions to estimators, from which confidence intervals and hypothesis tests are constructed.

It is shown that the number of moments of the optimal income process for which the LLN and CLT apply depend on the number of finite moments possessed by the productivity shock. For example, when the shock is lognormal—and all moments are finite—we have the remarkable conclusion that the LLN and the CLT hold for all moments of the income process.

1.1 Existing Literature

The problem considered in this paper is one of deducing stability of the state variable process from the model *primitives* and the restrictions imposed by optimizing behavior. Key references include Brock and Mirman [5], Mirman and Zilcha [17], Donaldson and Mehra [7], Stokey, Lucas and Prescott [23] and Hopenhayn and Prescott [11].

These studies contain many important contributions to the theory of stochastic growth. On the precise problem considered in this paper, however, we obtain a stronger form of stability (geometric ergodicity) without many of their assumptions, such as convex technology, bounded shocks and infinite marginal product of capital at zero. [2]

In the deterministic case, stability of optimal nonconvex planning problems

[2] Previously, Stachurski [21] relaxed the assumption that productivity shocks have compact support. Mitra and Roy [18] studied stability and instability of stochastic optimal planning problems with bounded shocks where technology may be nonconvex and marginal productivity at zero is not generally infinite. Although they do not consider ergodicity, they do give sufficient conditions on the primitives to avoid collapse of the process to the origin.

was studied by Skiba [20], Majumdar and Mitra [13], and Dechert and Nishimura [6], among others. More general studies of stochastic optimal growth under nonconvexities include Majumdar, Mitra and Nyarko [14] and Nishimura, Rudnicki and Stachurski [19].

In earlier research LLN and CLT results for stochastic optimal growth models were usually proved using restrictions on the support of the productivity shock (c.f., e.g., [23, Ch. 14]). This is because when Markov processes are ergodic and have compact state space they are typically geometrically ergodic, and geometric ergodicity is in turn associated with LLN and CLT. LLN and CLT results for some stochastic growth models without bounded shocks are given in Stachurski [22], but the assumptions on technology are too strict for the general Brock–Mirman problem. Evstigneev and Flåm [9] and Amir and Evstigneev [1] studied CLT related properties of competitive equilibrium economies.

There is of course much general research in economics on ergodicity and stability of Markov chains which does not deal directly with the problem considered here (i.e., inferring stability from model primitives and optimizing behavior). Classic studies include Mirman [16] and Futia [10]. A recent survey is Bhattacharya and Majumdar [2]. [3] They consider chains that satisfy a splitting condition, as well as those that are contracting on average. This paper advocates an alternative approach using Lyapunov functions.

The remainder of the paper is structured as follows. Section 2 formulates the problem. Section 3 states our results. Section 4 outlines the method of proof. Main proofs are given in Section 5. The proofs of some lemmas are deferred to an appendix.

2 The Model

The economy produces a single good, which can either be consumed or invested. When technology is convex we can and do assume the existence of a single social planner, who implements a state-contingent savings policy to maximize the discounted sum of expected utilities. When technology is not convex we study the planning problem (rather than competitive equilibria), again solved by a single agent.

At the start of time t, the agent observes income y_t, which is then divided between savings and consumption. Savings is added one-for-one to the existing capital stock. For simplicity we assume that depreciation is total: current

[3] See also related papers in this symposium, edited by the same authors.

savings and the capital stock k_t are identified. Labor is supplied inelastically; we normalize the total quantity to one.

After the time t investment decision is made a shock ε_t is drawn by nature and revealed to the agent. Production then takes place, yielding at the start of next period output

$$y_{t+1} = f(k_t)\,\varepsilon_t,$$

The sequence $(\varepsilon_t)_{t=0}^{\infty}$ is uncorrelated; f describes technology.

The production technology is smooth and as usual the derivative is eventually zero—the standard Inada assumption. On the other hand, it may not be convex:

Assumption 2.1 *The function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, continuously differentiable and satisfies $f(0) = 0$, $\underline{\lim}_{k\downarrow 0} f'(k) > 1$ and $\overline{\lim}_{k\uparrow\infty} f'(k) = 0$.*

A standard condition for stability in *stochastic* models is that $f'(0) = \infty$. We will be developing a tighter condition. In the meantime, however, one needs sufficient productivity at low investment to be sure that the optimal policy will be interior, and that the Euler equation does in fact hold. This is the purpose of requiring $\underline{\lim}_{k\downarrow 0} f'(k) > 1$.

To formalize uncertainty, let each random variable ε_t be defined on a fixed probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is the set of outcomes, $\mathscr{F}$ is the set of events, and $\mathbb{P}$ is a probability. The symbol $\mathbb{E}$ will denote integration with respect to $\mathbb{P}$.

Assumption 2.2 *The shock ε is distributed according to ψ, a density on $\mathbb{R}_+$. The density ψ is continuous and strictly positive on the interior of its domain. In addition, the moments $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(1/\varepsilon)$ are both finite for some $p \geq 1$.*

For example, the class of lognormal distributions satisfies Assumption 2.2 for every $p \in \mathbb{N}$. By definition, $\mathbb{P}\{a \leq \varepsilon_t \leq b\} = \int_a^b \psi(dz)$ for all a, b and t, where here and in all of what follows $\psi(dz)$ represents $\psi(z)dz$.

In earlier studies it was commonly assumed that the shock ε only took values in a closed interval $[a, b] \subset (0, \infty)$. In this case $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(1/\varepsilon)$ are automatically finite. For unbounded shocks the last two restrictions can be interpreted as bounds on the size of the right and left hand tails of ψ respectively. Without such bounds the stability of the economy is jeopardized.

The assumption that the shock distribution has a density representation which is continuous and everywhere positive is needed to complete the proof of geometric ergodicity. It is not related to the basic idea of the paper, and without it alternative stability results will be available. However global stability may fail—for example when technology is nonconvex and the shock is degenerate [6].

The larger p can be taken in Assumption 2.2, the tighter the conclusions of the paper will be. For example, we prove that the Law of Large Numbers holds for all moments of the optimal process up to order p, and the Central Limit Theorem holds for all moments up to order q, where $q \leq p/2$.

A feasible savings policy is a (Borel) function π from $\mathbb{R}_+$ to itself such that $0 \leq \pi(y) \leq y$ for all y. The set of all feasible policies will be denoted by Π. Corresponding to each $\pi \in \Pi$ there is a consumption policy $c^\pi(y) := y - \pi(y)$.

Every $\pi \in \Pi$ defines a Markov process on $(\Omega, \mathscr{F}, \mathbb{P})$ for income via

$$y_{t+1} = f(\pi(y_t))\, \varepsilon_t. \tag{1}$$

The problem for the agent is to choose a policy which solves

$$\max_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(c^\pi(y_t))\right], \tag{2}$$

where, for given π, the sequence $(y_t)_{t=0}^\infty$ is determined by (1). The number $\varrho \in (0,1)$ is the discount factor, and u is the period utility function.

Assumption 2.3 *The function* $u\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing, bounded, strictly concave, continuously differentiable, and* $\lim_{c \downarrow 0} u'(c) = \infty$.

2.1 Optimal policies

A policy π is called optimal if it is an element of Π and attains the maximum in (2). We are particularly interested in situations where optimal policies exist, are interior and satisfy the Euler equation. There are essentially two routes to the Euler equation. One is via convexity of the feasible set [17]. The second is to use smoothness of the productivity shock distribution [3,19]. Thus, Assumption 2.4:

Assumption 2.4 *Either* (i) *the production function* f *is concave; or* (ii) *the density* ψ *of the shock is continuously differentiable on* $(0, \infty)$*, and the integral* $\int z|\psi'(z)|dz$ *is finite.*

The class of lognormal shocks satisfies (ii) of Assumption 2.4.

Theorem 2.1 *Under Assumptions 2.1–2.3 there is at least one optimal policy for* (2)*, and every optimal policy is nondecreasing in income. If in addition Assumption 2.4 holds, then every optimal policy* π *is interior, and satisfies the*

Euler equation

$$u' \circ c^{\pi}(y) = \varrho \int u' \circ c^{\pi}[f(\pi(y))z] f'(\pi(y)) z \psi(dz), \quad \forall\, y \in (0, \infty).$$

Here $u' \circ c^{\pi}$ is of course the composition of u' and c^{π}, so that $u' \circ c^{\pi}(y)$ is the marginal utility of consumption when income equals y. By interiority is meant that $0 < \pi(y) < y$ for all $y \in (0, \infty)$.

Proof. For f concave the proof of Theorem 2.1 is well known [17,23]. For the other case (i.e., (ii) of Assumption 2.4), see [19, Lemma 3.1, Lemma 3.2, Proposition 3.1 and Proposition 3.2]. [4]

2.2 Dynamics

Once an initial condition for income is specified, each optimal policy completely defines the process $(y_t)_{t=0}^{\infty}$ for income via the recursion (1). Formally, $(y_t)_{t=0}^{\infty}$ is a Markov process on $(\Omega, \mathscr{F}, \mathbb{P})$. It is simplest in what follows to take the state space for the optimal process $(y_t)_{t=0}^{\infty}$ to be $(0, \infty)$ rather than $\mathbb{R}_+$. After all, each optimal policy is interior, and since the shock is distributed according to a density, $(y_t)_{t=0}^{\infty}$ remains in $(0, \infty)$ with probability one provided that $y_0 > 0$. We can always assume that $y_0 > 0$, as dynamics from $y_0 = 0$ are trivial.

Since each y_t is a random variable taking values in $(0, \infty)$ it has a distribution on the same. [5] A characteristic of Markov chains is that the sequence of distributions corresponding to the sequence of state variables satisfies a fundamental recursion, now to be described. [6]

To begin, let π be a fixed optimal policy, and let $\Gamma(y, \cdot)$ be the distribution for y_{t+1} given that $y_t = y$. Recall that if X is a real valued random variable with density f_X, if $a > 0$, and if $Y := a \cdot X$, then Y has density $f_Y(x) = f_X(x/a)(1/a)$. From this expression, the strict positivity of $f(\pi(y))$ for each y in the state space (Theorem 2.1) and (1) it follows that $\Gamma(y, \cdot)$ is a density given by

$$\Gamma(y, y') = \psi\left(\frac{y'}{f(\pi(y))}\right) \frac{1}{f(\pi(y))}. \tag{3}$$

[4] The assumption in [19] that the shock has mean one is not used in the proofs of these four results, and hence is omitted.
[5] Precisely, the distribution of y_t is the image measure $\mathbb{P} \circ y_t^{-1}$.
[6] More discussion of what follows can be found in Stokey, Lucas and Prescott [23], Stachurski [21] and Bhattacharya and Majumdar [2].

Now let $\mathscr{D}$ be the collection of densities on $(0, \infty)$.[7] Suppose for the moment that the initial condition y_0 is a random variable, with distribution equal to $\varphi_0 \in \mathscr{D}$. It then follows that the distribution of y_t is a density φ_t for all t, and the sequence $(\varphi_t)_{t=0}^{\infty}$ satisfies

$$\varphi_{t+1}(y') = \int \Gamma(y, y')\varphi_t(y)dy \tag{4}$$

for all $t \geq 0$. The intuition is that $\Gamma(y, y')$ is the probability of moving from income y to income y' in one period; in which case (4) simply states that the probability of being at y' next period is the probability of moving to y' via y, summed across all y, weighted by the probability that current income is equal to y.

It is perhaps more natural to regard y_0 as a single point, rather than a random variable with a density. In this case, provided $y_0 > 0$, one can take φ_0 to be the degenerate probability at y_0, and set $\varphi_1(\cdot) = \Gamma(y_0, \cdot) \in \mathscr{D}$. The remaining sequence of densities is then defined recursively by (4). Let us agree to write $\varphi_t^{y_0}$ for the t-th element so defined.

A $\varphi^* \in \mathscr{D}$ is called stationary for the optimal process (1) if it satisfies

$$\varphi^*(y') = \int \Gamma(y, y')\varphi^*(y)dy, \quad \forall\, y' \in (0, \infty). \tag{5}$$

It is clear from (4) and (5) that if y_t has distribution φ^*, then so does y_{t+n} for all $n \in \mathbb{N}$. A density satisfying (5) is also called a stochastic steady state. At such a long-run equilibrium the *probabilities* are stationary over time, even though the state variable is not.[8]

Finally we define ergodicity, which is the fundamental result of Brock and Mirman [5]. Ergodicity of the optimal process (1) means that there is a unique $\varphi^* \in \mathscr{D}$ satisfying (5), and, moreover,

$$\|\varphi_t^y - \varphi^*\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{for all } y > 0,$$

where $\| \cdot \|$ is the L_1 norm.[9] Geometric ergodicity essentially means that the process is ergodic and, in addition, $\|\varphi_t^y - \varphi^*\| = O(\alpha^t)$ for some $\alpha < 1$. A more precise definition will be given in the statement of results.

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[7] To be precise, $\mathscr{D}$ is the set of nonnegative Borel functions on $(0, \infty)$ that integrate to one.
[8] Why have we not defined stationary *distributions*, which are more general than stationary densities? The answer is that for our model all stationary distributions will in fact be densities. A proof is available from the authors.
[9] That is, $\|\varphi_t - \varphi^*\| = \int |\varphi_t - \varphi^*|$. Some studies use a weaker topology.

3 Results

Our main stability results are now presented. All of Assumptions 2.1–2.4 are maintained without further comment.

For the remainder of the paper let $\pi \in \Pi$ be a fixed optimal savings policy. As before, c^π is the corresponding consumption policy. Define

$$V(y) := \sqrt{u' \circ c^\pi(y)} + y^p + 1, \quad p \text{ as in Assumption 2.2.} \tag{6}$$

In the proofs V will play the role of Lyapunov function.

We now state our main result. For an outline of the proof see Section 4. A full proof is given in Section 5.

Theorem 3.1 *Let V be the function defined in* (6)*, and let $f'(0) := \underline{\lim}_{k \downarrow 0} f'(k)$. If the inequality*

$$f'(0) > \frac{\mathbb{E}(1/\varepsilon)}{\varrho} \tag{7}$$

holds, then the process $(y_t)_{t=0}^\infty$ defined by (1) *is geometrically ergodic. Precisely, $(y_t)_{t=0}^\infty$ has a unique stationary distribution φ^*, and, moreover, there is a constant $\alpha < 1$ and an $R < \infty$ such that*

$$\|\varphi_t^y - \varphi^*\| \leq \alpha^t R V(y), \quad \forall\, y > 0, \;\; \forall\, t \geq 0.$$

In (7) the term $\mathbb{E}(1/\varepsilon) = \int (1/z)\psi(dz)$ will be large when unfavorable shocks are likely. Note that if the shock is degenerate and puts all probability mass on one, then the stability condition reduces to $f'(0) > 1/\varrho$, which is the usual deterministic condition for stability. [10]

Now let h be a real function on the state space, and define $S_n(h) := \sum_{t=1}^n h \circ y_t$. The next result is to some extent a corollary of Theorem 3.1.

Theorem 3.2 *Let* (7) *hold, and let φ^* be the unique stationary distribution for the optimal process $(y_t)_{t=0}^\infty$. If $|h| \leq V$, then the Law of Large Numbers holds for h. That is,*

$$\mathbb{E}_{\varphi^*}(h) := \int h \, d\varphi^* < \infty, \quad \textit{and} \quad \lim_{n \to \infty} \frac{S_n(h)}{n} = \mathbb{E}_{\varphi^*}(h) \;\; \mathbb{P}\text{-}a.s. \tag{8}$$

If in addition $h^2 \leq V$, then the Central Limit Theorem also holds for h.

[10] Kamihigashi [12, Theorem 3.1] provides a partial converse to Theorem 3.1. He shows that $y_t \to 0$ holds $\mathbb{P}$-a.s. whenever $f'(0) < \exp(-\mathbb{E} \ln \varepsilon)$.

Precisely, there is a constant $\sigma^2 \in \mathbb{R}_+$ such that

$$\frac{S_n(h - \mathbb{E}_{\varphi^*}(h))}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2). \tag{9}$$

In the statement of the theorem the symbol $\xrightarrow{d}$ means convergence in distribution. If $\sigma^2 = 0$ then the right hand side of (9) is interpreted as the probability measure concentrated on zero. Also, $(h - \mathbb{E}_{\varphi^*}(h))(y_t) := h(y_t) - \mathbb{E}_{\varphi^*}(h)$.

These results are in a rather convenient form. In particular, since $x^p \leq V(x)$ we see that all moments of the income process up to order p satisfy the LLN, and all moments up to order q satisfy the CLT, where q is the largest integer such that $2q \leq p$.

4 Outline of Techniques

In this section we describe the main ideas used in the proof. (The details are in Section 5.) First the Foster–Lyapunov approach to stability of Markov chains is outlined. As in deterministic dynamical systems, Lyapunov methods often provide the most readily applicable techniques for stability proofs. [11]

In essence, Lyapunov functions are constructed to assign large values to the "edges" of the state space, and small values to the center. If one can show using the law of motion for the system that the value the Lyapunov function assigns to the next-period state is expected to contract relative to the value assigned to the current state, then it must be that the state is moving away from the edges of the state space and towards the center. This behavior is associated with stability.

However, the choice of appropriate Lyapunov function and proof of the above mentioned contraction property is rarely trivial. The main task, then, is to find an appropriate function and establish the contraction condition. We suggest using marginal utility of consumption as a Lyapunov function, and the Euler equation as the contraction condition. In short, the idea is that for low levels of income marginal returns to investment are likely to be large, which via the Euler equation requires a high marginal willingness to substitute future for current consumption. In other words, expected marginal utility of consumption (the value of the Lyapunov function) for the next period state is small relative to that of the current state. This provides the contraction property.

[11] In the case of Markov chains they are very general too—existence of Lyapunov functions satisfying contraction conditions often characterize stability properties.

Actually it is more accurate to say that marginal utility forms a component of our Lyapunov function. It is large at the "edge" of the state space near the origin, but not near the other edge, at plus infinity. Hence contraction with respect to marginal utility only means that the state does not get too small. So another component to the Lyapunov function (which is large at plus infinity) will have to be spliced on. The mechanics are detailed below.

We begin with a general discussion of Foster–Lyapunov methods. The usual way to implement the notion that a potential Lyapunov function V takes small values in the center of the state space and is large towards the edges is to require that all its sublevel sets be precompact. Here precompactness means having compact closure, and sublevel sets of V are sets of the form $\{x : V(x) \leq a\}$ for real valued a.

Definition 4.1 *A Lyapunov function on a topological space S is a nonnegative real function on S with the property that all sublevel sets are precompact.*

Lemma 4.1 *If $S = (0, \infty)$, then $V \colon S \to \mathbb{R}_+$ is a Lyapunov function if and only if* $\lim_{x \downarrow 0} V(x) = \lim_{x \uparrow \infty} V(x) = \infty$. [12]

The proof is straightforward and we omit it. From Lemma 4.1 and the fact that $u' \circ c^\pi \geq u'$ it is easy to see that V defined in (6) is a Lyapunov function.

Suppose now that for some Lyapunov function V on $(0, \infty)$ and constants $\lambda < 1$ and $b < \infty$ we have

$$\int V[f(\pi(y))z]\psi(dz) \leq \lambda V(y) + b, \quad \forall y \in (0, \infty). \tag{10}$$

This in essence says that when the value assigned by V to the state is large (i.e., when current income y is either close to zero or very large), the value assigned to the next period state is expected to be less than the current value (loosely speaking, income moves back towards the center of the state space). [13]

To gain some understanding of the implications of (10), recall that (by the Markov property)

$$\mathbb{E}[V(y_{t+1}) \,|\, y_t] = \int V[f(\pi(y_t))z]\psi(dz) \quad \mathbb{P}\text{-a.s.}$$

$$\therefore \quad \mathbb{E}[V(y_{t+1}) \,|\, y_t] \leq \lambda V(y_t) + b \quad \mathbb{P}\text{-a.s.}$$

Taking expectations of both sides and using the law of iterated expectations gives $\mathbb{E}V(y_{t+1}) \leq \lambda \cdot \mathbb{E}V(y_t) + b$. Extrapolating this inequality from time zero

[12] We stress that on $(0, \infty)$ topological concepts such as precompactness always refer to the *relative* Euclidean topology.

[13] To see this, write (10) as $\int V[f(\pi(y))z]\psi(dz)/V(y) \leq \lambda + b/V(y)$. Since $\lambda < 1$ and $b < \infty$ clearly $\int V[f(\pi(y))z]\psi(dz) < V(y)$ for sufficiently large $V(y)$.

(where y_0 is a given constant) forward to time t and using the fact that $\lambda < 1$ gives the bound

$$\mathbb{E}V(y_t) \le V(y_0) + \frac{b}{1-\lambda} =: M, \quad \forall\, t \in \mathbb{N}. \tag{11}$$

$$\therefore \quad \mathbb{P}\{V(y_t) > n\} \le \frac{M}{n}, \quad \forall\, t, n \in \mathbb{N} \qquad (\because \text{ Chebychev's ineq.})$$

Thus for each $n \in \mathbb{N}$ there is a compact $K_n \subset (0, \infty)$ containing $\{x : V(x) \le n\}$ with

$$\int_{K_n^c} \varphi_t^{y_0}(y)dy = \mathbb{P}\{y_t \notin K_n\} \le \frac{M}{n}, \quad \forall\, t \in \mathbb{N},$$

where $K_n^c := K_n \setminus (0, \infty)$. [14] It is now evident that

$$\forall \varepsilon > 0, \;\; \exists \text{ a compact } K \subset (0, \infty) \;\text{ s.t. } \sup_{t \in \mathbb{N}} \int_{K^c} \varphi_t^{y_0}(y)dy < \varepsilon.$$

This statement says precisely that (for any given starting point y_0) the sequence of distributions $(\varphi_t^{y_0})_{t=0}^{\infty}$ generated by (4) is *tight*. Tightness is very closely linked with many stability properties. But even without invoking these results one can see directly that apart from an arbitrarily small ε, all probability mass for the sequence $(y_t)_{t=0}^{\infty}$ stays on a compact subset of $(0, \infty)$, in which case it cannot be escaping to zero or plus infinity.

One way to construct a Lyapunov function satisfying (10) is to take a pair of real functions w_1 and w_2 with the properties $\lim_{x \downarrow 0} w_1(x) = \infty$, $\lim_{x \uparrow \infty} w_2(x) = \infty$, and

$$\int w_i[f(\pi(y))z]\psi(dz) \le \lambda_i w_i(y) + b_i, \quad \forall y \in (0, \infty); \quad i = 1, 2 \tag{12}$$

for some $\lambda_1, \lambda_2, b_1, b_2$ with $\lambda_i < 1$ and $b_i < \infty$. In this case it is easy to show that

Proposition 4.1 *If $V := w_1 + w_2 + 1$, where w_1 and w_2 are two real functions on $(0, \infty)$ satisfying (12), and $\lim_{x \downarrow 0} w_1(x) = \lim_{x \uparrow \infty} w_2(x) = \infty$, then V is a Lyapunov function on $(0, \infty)$ satisfying (10) for $\lambda := \max\{\lambda_1, \lambda_2\}$ and $b := b_1 + b_2 + 1$.* [15]

Using Proposition 4.1 we will show that V defined in (6) is a Lyapunov function satisfying (10) for some $\lambda < 1$ and $b < \infty$. The advantage of constructing V from two components w_1 and w_2 is that we can treat the two problems of diverging to zero (economic collapse) and diverging to infinity (unbounded growth) separately. Obtaining a contraction with respect to w_1 prevents the

[14] Such a compact set exists by the definition of V.
[15] The reason for adding 1 to $w_1 + w_2$ in the definition of V will become clear in the proof of Proposition 5.1

former (because w_1 is large at zero), while obtaining one for w_2 prevents the latter (because w_2 is large at infinity).

When diminishing returns are present, eliminating the possibility of unbounded growth is straightforward. The reason is that unbounded growth cannot occur for *any* feasible savings policy—even one that invests all output. There is no need to deal with the subtleties of optimization and the Euler equation.

Eliminating the possibility of collapse—finding a suitable function w_1 satisfying a contraction in the form of (12)—is considerably more difficult. The question is whether or not economies invest sufficiently to sustain a nontrivial long-run equilibrium. This depends on preferences (particularly rates of time discount), returns to investment, and the distribution of the productivity shock.

Our method for finding a suitable w_1 satisfying (12) is to use the Euler equation directly. This leads to the most important result of the paper (the rest is something of a mopping up operation):

Proposition 4.2 *Let $w_1 := \sqrt{u' \circ c^\pi}$. If the inequality* (7) *is satisfied, then there exists a $\lambda_1 < 1$ and a $b_1 < \infty$ such that*

$$\int w_1[f(\pi(y))z]\psi(dz) \leq \lambda_1 w_1(y) + b_1, \quad \forall y \in (0, \infty). \tag{13}$$

Proof. We will make use of the fact that if g and h are positive real functions on $(0, \infty)$, then, by the Cauchy–Schwartz inequality,

$$\int (gh)^{1/2} d\psi \leq \left(\int g \, d\psi \cdot \int h \, d\psi \right)^{1/2}. \tag{14}$$

Recall the Euler equation:

$$u' \circ c^\pi \,(y) = \varrho f'(\pi(y)) \int u' \circ c^\pi [f(\pi(y))z] z \psi(dz). \tag{15}$$

Set $w_1 := \sqrt{u' \circ c}$, as in the statement of the proposition. From (14) we have

$$\begin{aligned} \int w_1[f(\pi(y))z]\psi(dz) &= \int [u' \circ c^\pi (f(\pi(y))z) z (1/z)]^{1/2} \psi(dz) \\ &\leq \left(\int u' \circ c^\pi [f(\pi(y))z] z \psi(dz) \cdot \int (1/z) \psi(dz) \right)^{1/2}. \end{aligned}$$

Combining this with (15) now gives

$$\int w_1[f(\pi(y))z]\psi(dz) \leq \left[\frac{\mathbb{E}(1/\varepsilon)}{\varrho f'(\pi(y))} \right]^{1/2} w_1(y).$$

From (7) one can deduce the existence of a $\delta > 0$ and a $\lambda_1 \in (0,1)$ such that

$$\left[\frac{\mathbb{E}(1/\varepsilon)}{\varrho f'(\pi(y))}\right]^{1/2} < \lambda_1 < 1, \quad \forall\, y < \delta.$$

$$\therefore \quad \int w_1[f(\pi(y))z]\psi(dz) \leq \lambda_1 w_1(y), \quad \forall y < \delta. \tag{16}$$

We also need the following technical bound, the proof of which is given in the appendix.

Lemma 4.2 *Given δ in* (16)*, there exists a $b_1 < \infty$ such that*

$$\int w_1[f(\pi(y))z]\psi(dz) \leq b_1, \quad \forall y \geq \delta. \tag{17}$$

From (16) and (17) the inequality (13) is immediate.

We now establish the complementary result for w_2.

Proposition 4.3 *Let p be as in Assumption 2.2. For w_2 defined by $w_2(y) = y^p$, there exists a $\lambda_2 < 1$ and a $b_2 < \infty$ such that*

$$\int w_2[f(\pi(y))z]\psi(dz) \leq \lambda_2 w_2(y) + b_2, \quad \forall y \in (0, \infty). \tag{18}$$

Proof. First choose $\gamma \in (0,1)$ so that $\gamma^p \mathbb{E}(\varepsilon^p) < 1$. For such a γ it follows from the assumption $\overline{\lim}_{k\to\infty} f'(k) = 0$ that we can find a $d < \infty$ such that whenever $y > d$ we have $f(y) \leq \gamma \cdot y$.[16] For all $y \in (0, d]$ we have $f(\pi(y)) \leq f(y) \leq f(d)$, and hence

$$\int [f(\pi(y))z]^p \psi(dz) \leq f(d)^p \mathbb{E}(\varepsilon^p), \quad \forall y \leq d.$$

On the other hand, $y > d$ implies $f(\pi(y)) \leq f(y) \leq \gamma y$, so

$$\int [f(\pi(y))z]^p \psi(dz) \leq \gamma^p \mathbb{E}(\varepsilon^p) y^p, \quad \forall y > d.$$

Setting $\lambda_2 := \gamma^p \mathbb{E}(\varepsilon^p)$ and $b_2 := f(d)^p \mathbb{E}(\varepsilon^p)$ gives (18).

Summarizing Proposition 4.1, Proposition 4.2 and Proposition 4.3,

[16] To see this, choose $\bar{y}$ such that $y \geq \bar{y}$ implies $f'(y) \leq \gamma/2$. By the Fundamental Theorem of Calculus, when $y \geq \bar{y}$ we have $f(y) \leq f(\bar{y}) + (y - \bar{y})(\gamma/2)$. This function in turn is dominated by γy for y sufficiently large—larger than $\hat{y}$, say. Now set $d := \max\{\bar{y}, \hat{y}\}$.

Proposition 4.4 *The function V defined in* (6) *is a Lyapunov function on* $(0, \infty)$*, and under the hypotheses of Theorems 3.1 and 3.2 the contraction condition* (10) *holds for some $\lambda < 1$ and $b < \infty$.*

5 Proofs

The complete proofs of Theorems 3.1 and 3.2 are now given. They center on establishing that the optimal process is V-uniformly ergodic for V specified by (6), where V-uniform ergodicity is defined in Meyn and Tweedie [15, Ch. 16]. Essentially this requires geometric drift towards a subset of the state space which satisfies a certain minorization condition.

The following definitions are necessary. Let $\mathscr{B}$ be the Borel sets on $(0, \infty)$, let $\mathscr{M}$ be the finite measures on $((0, \infty), \mathscr{B})$, and let $\mathscr{P}$ be all $\nu \in \mathscr{M}$ with $\nu(0, \infty) = 1$. For $B \in \mathscr{B}$ let $\mathbb{1}_B$ denote the indicator function of B. Proofs of lemmas are given in the appendix.

Definition 5.1 *Let $\mu \in \mathscr{P}$. The optimal process* (1) *is called μ-irreducible if*

$$\mathbb{P}\{y_t \in B \text{ for some } t \in \mathbb{N}\} > 0, \quad \forall\, y_0 > 0,\ B \in \mathscr{B} \text{ with } \mu(B) > 0.$$

In other words, $(y_t)_{t=0}^{\infty}$ visits every set of positive μ-measure from every starting point.

Definition 5.2 *A set $C \in \mathscr{B}$ is called a C-set for the optimal process* (1) *if there is a nontrivial $\nu \in \mathscr{M}$ such that*

$$y \in C \implies \left\{ \int_B \Gamma(y, y')dy' \geq \nu(B), \quad \forall\, B \in \mathscr{B} \right\}, \tag{19}$$

where Γ is the stochastic kernel defined in (3).

Definition 5.3 *The optimal process is called strongly aperiodic if* (19) *holds for some $C \in \mathscr{B}$ and nontrivial $\nu \in \mathscr{M}$ (i.e., Γ has a C-set), and, moreover, $\nu(C) > 0$.*

Definition 5.4 *Let V be as in* (6)*. The optimal process $(y_t)_{t=0}^{\infty}$ is called V-uniformly ergodic* [15, Ch. 16] *if*

$$\sup_{y>0} \left\{ \frac{\|\varphi_t^y - \varphi^*\|}{V(y)} \right\} \to 0 \text{ as } t \to \infty.$$

Almost all the results derived in this paper follow from

Proposition 5.1 *Under the hypotheses of Theorems 3.1 and 3.2, the optimal process* $(y_t)_{t=0}^\infty$ *is* V*-uniformly ergodic.*

Proof. By Meyn and Tweedie [15, Theorem 16.1.2], the optimal process will be V-uniformly ergodic whenever it is μ-irreducible for some $\mu \in \mathscr{P}$, strongly aperiodic, and there is a C-set $C \in \mathscr{B}$, an $\alpha < 1$, a $\beta < \infty$ and a real function $V\colon (0,\infty) \to [1,\infty)$ such that

$$\int V[f(\pi(y))z]\psi(dz) \le \alpha V(y) + \beta \mathbb{1}_C(y), \quad \forall y \in (0,\infty). \tag{20}$$

(Actually the theorem only requires aperiodicity rather than strong aperiodicity, and that C be "petite," which is a generalization of the usual notion of a C-set.)

The following three lemmas are all proved in the appendix.

Lemma 5.1 *The optimal process is* μ*-irreducible for every* $\mu \in \mathscr{P}$ *which is absolutely continuous with respect to Lebesgue measure.*

Lemma 5.2 *Every compact subset of the state space* $(0,\infty)$ *is a* C*-set.*

Lemma 5.3 *The optimal process is strongly aperiodic.*

As a result of Lemmas 5.1–5.3, to establish Proposition 5.1 it is sufficient to show that there is an $\alpha < 1$, a $\beta < \infty$ and a *compact* set C such that (20) holds for V defined in (6).

Let λ and b be as in Proposition 4.4. Set $\beta := b$. Take any α such that $\lambda < \alpha < 1$. By the Lyapunov property, we can choose a compact set $C \subset (0,\infty)$ such that $V(y) \ge \beta/(\alpha - \lambda)$ whenever $y \notin C$. For $y \in C$ the bound (20) is trivial, because by Proposition 4.4

$$\int V[f(\pi(y))z]\psi(dz) \le \lambda V(y) + b \le \alpha V(y) + \beta \mathbb{1}_C(y).$$

For $y \notin C$ the bound (20) also holds, because

$$\frac{\int V[f(\pi(y))z]\psi(dz)}{V(y)} \le \lambda + \frac{b}{V(y)} \le \alpha.$$

Finally, $V \ge 1$ by construction. This completes the proof of Proposition 5.1.

Proof of Theorem 3.1 Immediate from Proposition 5.1 and [15, Theorem 16.0.1, Part (ii)].

Proof of Theorem 3.2 LLN Result: By Meyn and Tweedie [15, Theorem 17.0.1, Part (i)], the LLN holds for h provided that $(y_t)_{t=0}^{\infty}$ is positive Harris and $\int |h| d\varphi^* < \infty$. By positive Harris is meant that $(y_t)_{t=0}^{\infty}$ has an invariant distribution and is Harris recurrent. For a definition of Harris recurrence see Meyn and Tweedie [15, Ch. 9]. For our purposes we need only note that by the same reference, Proposition 9.1.8, Harris recurrence holds when the sublevel sets of V are all C-sets, and, in addition, that there is a C-set $C \in \mathscr{B}$ such that

$$\int V[f(\pi(y))z]\psi(dz) \leq V(y), \quad \forall y \notin C. \tag{21}$$

We have already shown that sublevel sets of V are C-sets in Lemma 5.2 (recall that sublevel sets of V are precompact, and that measurable subsets of C-sets are C-sets). That there is a C-set satisfying (21) is immediate from (20).

Therefore $(y_t)_{t=0}^{\infty}$ is positive Harris, and it remains only to show that $\int |h| d\varphi^* := \int |h(y)|\varphi^*(y)dy < \infty$. Since $|h| \leq V$, it is sufficient to show that $\int V d\varphi^*$ is finite.

To see that this is the case, pick any initial condition y_0. For $n \in \mathbb{N}$ let $K_n := \mathbb{1}_{[1/n,n]}$. By (11) there is an $M < \infty$ satisfying $\int V d\varphi_t^{y_0} \leq M$ for all t, and hence $\int K_n V d\varphi_t^{y_0} \leq M$ for all t and n.

Lemma 5.4 *The function V is bounded on compact sets.*

From this lemma we see that $K_n V$ is bounded, so (since L_1 convergence implies weak convergence) taking the limit with respect to t gives $\int K_n V d\varphi^* \leq M$ for all n. Now taking limits with respect to n and using Monotone Convergence gives $\int V d\varphi^* < \infty$.

CLT Result: Immediate from Meyn and Tweedie [15, Theorem 17.0.1, Parts (ii)–(iv)] and Proposition 5.1. The constant σ^2 is given by

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{\varphi^*}[(S_n(h - \mathbb{E}_{\varphi^*}(h))^2].$$

A Appendix

Proof of Lemma 4.2 If part (i) of Assumption 2.4 holds (f is concave), then c^π is increasing [17]. We already have the bound

$$\int w_1[f(\pi(y))z]\psi(dz) \leq \left(\int u' \circ c^\pi[f(\pi(y))z]z\psi(dz) \cdot \int (1/z)\psi(dz)\right)^{1/2}$$

for all y. Since $u' \circ c^\pi$ is decreasing, we can set

$$b_1 := \left(\int u' \circ c^\pi [f(\pi(\delta))z] z \psi(dz) \cdot \int (1/z) \psi(dz) \right)^{1/2}.$$

Suppose on the other hand that (ii) of Assumption 2.4 holds. Let $v\colon (0, \infty) \to \mathbb{R}$ be the value function for the dynamic programming problem. That is,

$$v(y) = \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(c^\pi(y_t)) \right] \quad \text{when } y_0 = y.$$

Let $r > 0$. By a change of variable we can write

$$\int_0^\infty v(rz)\psi(z)dz = \int_{-\infty}^{\infty} v[\exp(\ln r + x)]\psi(e^x)e^x \, dx.$$

Define

$$\mu(r) := \int_{-\infty}^{\infty} h(x + r) g(x) dx,$$

where $h(y) = v[\exp(y)]$ and $g(y) = \psi(e^y)e^y$, so that when μ is differentiable we have

$$\frac{d}{dr} \int_0^\infty v(rz)\psi(dz) = \frac{1}{r}\mu'(\ln r).$$

By Nishimura, Rudnicki and Stachurski [19, Lemma 5.1], μ is continuously differentiable, and

$$\mu'(\ln r) = -\int_{-\infty}^{\infty} h(x + \ln r) g'(x) \, dx = \int_{-\infty}^{\infty} h'(x + \ln r) g(x) \, dx, \tag{A.1}$$

where h' is defined as the derivative of h where it exists and zero elsewhere. Reversing the change of variables gives

$$\int_{-\infty}^{\infty} h'(x + \ln r) g(x) \, dx = r \int_0^\infty v'(rz) z \psi(dz). \tag{A.2}$$

Combining (A.1) and (A.2) gives

$$\int_0^\infty v'(rz) z \psi(dz) = \frac{1}{r}\mu'(\ln r) = -\frac{1}{r}\int_{-\infty}^{\infty} h(x + \ln r) g'(x) \, dx.$$

Therefore, using the fact that v is bounded by a constant M, say (because u is bounded: Assumption 2.3),

$$\begin{aligned}
\int_0^\infty v'(rz) z \psi(dz) &\leq \frac{1}{r}\int_{-\infty}^{\infty} |h(x + \ln r) g'(x)| \, dx \\
&= \frac{1}{r}\int_{-\infty}^{\infty} v(\exp(x + \ln r)) |\psi'(e^x)e^{2x} + \psi(e^x)e^x| \, dx \\
&\leq \frac{M}{r}\int_{-\infty}^{\infty} |\psi'(e^x)e^{2x} + \psi(e^x)e^x| \, dx \\
&\leq \frac{M}{r}\int_0^\infty [|\psi'(z)|z + \psi(z)] \, dz.
\end{aligned}$$

In light of Assumption 2.4, part (ii), then, there is a constant N such that

$$\int_0^\infty v'(f(\pi(y))z)z\psi(dz) \leq \frac{N}{f(\pi(y))}.$$

Given δ as in the statement of the lemma, and using the fact that $u' \circ c^\pi$ is equal to v' almost everywhere [19, Proposition 3.1, parts 3 & 4], we have the bound

$$\int_0^\infty u' \circ c^\pi [f(\pi(y))z] z\psi(dz) \leq b_0 := \frac{N}{f(\pi(\delta))}, \quad \forall y \geq \delta.$$

But then, applying Cauchy–Schwartz again,

$$\int_0^\infty w_1[f(\pi(y))z]\psi(dz) \leq b_1 := [b_0 \mathbb{E}(1/\varepsilon)]^{1/2}, \quad \forall y \geq \delta.$$

Proof of Lemma 5.1 Let $q \in \mathscr{D}$. Take any $B \in \mathscr{B}$ with positive q-measure and any $y_0 \in (0, \infty)$. It is easy to check that the set $[f(\pi(y_0))]^{-1} \cdot B$ has positive Lebesgue measure. We have $\mathbb{P}\{y_1 \in B\} = \int_{\{z : f(\pi(y_0))z \in B\}} \psi(dz) = \int_{[f(\pi(y_0))]^{-1} \cdot B} \psi(z)dz$. The latter is strictly positive by the strict positivity of ψ (Assumption 2.2).

Proof of Lemma 5.2 Measurable subsets of C-sets are easily seen to be C-sets. Evidently then it is sufficient to establish that the interval $C_k := [1/k, k]$ is a C-set for every $k \in \mathbb{N}$. Pick any $k \in \mathbb{N}$. By interiority and monotonicity of the optimal policy (Theorem 2.1) we have

$$0 < f(\pi(1/k)) \leq f(\pi(y)) \leq f(\pi(k)) < \infty, \quad \forall y \in C_k.$$

Since ψ is continuous and strictly positive it follows that

$$\inf_{C_k \times C_k} \Gamma(y, y') = \inf_{C_k \times C_k} \psi\left(\frac{y'}{f(\pi(y))}\right) \frac{1}{f(\pi(y))} =: r > 0.$$

Now letting ν be the measure defined by $\nu(B) = r \cdot \int_B \mathbb{1}_{C_k}(x)dx$ gives (19) for all $y \in C_k$.

Proof of Lemma 5.3 Clearly $\nu(C_k) > 0$ holds for the set C_k and measure ν given in the proof of Lemma 5.2.

Proof of Lemma 5.4 If Assumption 2.4 part (i) holds (f concave), then $u' \circ c$ is continuous [17] and the result is clear. Suppose instead that Assumption 2.4 part (ii) holds. In that case $u' \circ c$ is again bounded on compacts, this time by Nishimura, Rudnicki and Stachurski [19, Lemma 5.3].

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