Solving Recursive Utility Models with Preference Shocks

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(with thanks to Chase Coleman and Pablo Levi)

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Scope

- Focus entirely on Epstein-Zin preferences
- Applications are all in asset pricing
- Seek conditions for existence and uniqueness of solutions
- Conditions are necessary as well as sufficient
- Globally convergent solution methods
- Implementation on GPUs

Related Work

Pohl, Schmedders and Wilms (2018, JF)

- full solutions using projection methods
- shows value of treating original nonlinear models
- no existence / uniqueness / global convergence results

Bloise and Vailakis (2018, JET)

- valuable DP results in a recursive setting
- uses concave monotone operator methods
- no preference shocks
- sufficient but not necessary conditions

Related Work

See also

- Epstein and Zin (1989, ECMA)
- Le Van and Vailakis (2005, JET)
- Marinacci and Montrucchio (2010, JET)
- Hansen and Scheinkman (2012, PNAS)
- Christensen (2021, working paper)

Closest Related Work

Borovicka and Stachurski (2020, JF)

- ignores preference shocks

Stachurski and Zhang (2021, JET)

- restricted parameter values
- restricted preference shocks
- sufficient but not necessary conditions
- no global convergence results

Asset Pricing Background

Pricing a claim to a **cash flow** $\{D_t\}$ via

$$P_t = \mathbb{E}_t M_{t+1}(D_{t+1} + P_{t+1}) \tag{1}$$

- $\{M_t\}$ = **stochastic discount factor** (SDF) process

Example. In Lucas (1978),

$$M_t = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

Example. Mehra and Prescott (1985) apply this SDF CRRA with u

Important: $\{M_t\}$ can be used to price a claim to **any** cash flow

- dividend stream from holding PepsiCo shares
- constant cash flow from risk-free bond
- cash flow from holding one Dogecoin?

A tough ask, which the Lucas SDF fails (e.g., risk premium puzzle)

We need some more free parameters!

One line of approach:

- Epstein–Zin preferences
- with preference shocks!

Epstein–Zin preferences defined recursively by

$$V_t = \left[(1-\beta)\lambda_t C_t^{1-1/\psi} + \beta \left\{ \mathcal{R}_{t,1-\gamma}\left(V_{t+1}\right) \right\}^{1-1/\psi} \right]^{1/(1-1/\psi)}$$

A popular specification in quantitative finance

- Albuquerque et al. (2016, JF)
- Schorfheide, Song and Yaron (2018, ECMA)
- Gomez-Cram and Yaron (2020, RFS)
- etc.

Before working through this, let's go back a few steps

- What's different about recursive preference models?
- How should we solve them?
- How does this change when we add preferences shocks?

Recursive Preferences Background

Let's value a Markov process $\{X_t\}$ with

$$\mathbb{P}\{X_{t+1} \in B \mid X_t = x\} = \int_B q(x, y)\, \mathrm{d}y$$

Current reward from state X_t is $r(X_t)$

Example. Valuing a consumption stream

- $C_t = g(X_t)$
- utility is $u(C_t)$

Set $r = u \circ g$, so that $r(X_t) = u(g(X_t)) = u(C_t)$

Examples

⋆ **Classic linear aggregator**

$$v(x) = r(x) + \beta \int v(y) q(x, y) \, \mathrm{d}y \tag{2}$$

- discount factor $\beta \in (0, 1)$
- the **value function** v evaluates x given (r, β, q)

Sequential version is

$$v(x) = \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t r(X_t) \,|\, X_0 = x \right]$$

★ **Linear aggregator with preference shocks**

$$v(x) = r(x) + \beta(x) \int v(y) q(x, y) \, \mathrm{d}y \tag{3}$$

- now discounting is state dependent

Sequential version is

$$v(x) = \mathbb{E} \left\{ \sum_{t=0}^{\infty} \left[\prod_{i=0}^{t-1} \beta(X_i) \right] r(X_t) \mid X_0 = x \right\}$$

⋆ **CES aggregator**

$$v(x) = \left\{ r(x)^{1-1/\psi} + \beta \left[\int v(y) q(x,y) \, \mathrm{d}y \right]^{1-1/\psi} \right\}^{\frac{1}{1-1/\psi}}$$

- $\psi \neq 1$ measures elasticity of substitution

Sequential version is

$$\ldots ?$$

⋆ Epstein–Zin preferences

$$v(x) = \left\{ r(x)^{1-1/\psi} + \beta \left[\int v(y)^{1-\gamma} q(x,y) \, \mathrm{d}y \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}$$

- $\psi \neq 1$ measures elasticity of substitution
- $\gamma \neq 1$ measures risk aversion

Sequential version is

$$\ldots?$$

Solving with Linear Aggregators

Consider again

$$v(x) = r(x) + \beta \int v(y) q(x, y) \, \mathrm{d}y \tag{4}$$

Fixed point problem is

$$Tv(x) = r(x) + \beta \int v(y) q(x, y) \, \mathrm{d}y \tag{5}$$

$$|Tv(x) - Tw(x)| \leqslant \beta \int |v(y) - w(y)| \, q(x, y) \, \mathrm{d}y$$

Bounded case: for all x,

$$
\begin{aligned}
|Tv(x) - Tw(x)| &\leqslant \beta \int |v(y) - w(y)| \; q(x, y) \, \mathrm{d}y \\
&\leqslant \beta \int \|v - w\|_\infty \, q(x, y) \, \mathrm{d}y \\
&= \beta \|v - w\|_\infty
\end{aligned}
$$

$$
\therefore \quad \|Tv - Tw\|_\infty \leqslant \beta \|v - w\|_\infty
$$

Now use Banach

Unbounded case, where

$$\pi(y) = \int q(x, y)\pi(x)\,\mathrm{d}x$$

Integrate

$$|Tv(x) - Tw(x)| \leqslant \beta \int |v(y) - w(y)|\, q(x, y)\,\mathrm{d}y$$

to get

$$\int |Tv(x) - Tw(x)|\pi(x)\,\mathrm{d}x \leqslant \beta \int \int |v(y) - w(y)| q(x, y)\,\mathrm{d}y \pi(x)\,\mathrm{d}x$$

$$= \beta \int |v(y) - w(y)|\pi(y)\,\mathrm{d}y$$

Now use Banach in $L_1(\pi)$

Linear aggregator with preference shocks, where

$$v(x) = r(x) + \beta(x) \int v(y) q(x, y) \, \mathrm{d}y \tag{6}$$

Not always a one-step contraction

For example, in the bounded case, we get

$$\|Tv - Tw\|_\infty \leqslant \sup_x \beta(x) \|v - w\|_\infty$$

But, in many applications,

$$\mathbb{P}\{\beta(X_t) > 1\} > 0$$

How else can we handle

$$v(x) = r(x) + \beta(x) \int v(y) q(x, y) \, \mathrm{d}y? \tag{7}$$

Actually, it's easy: define K via

$$K g(x) = \beta(x) \int g(y) q(x, y)$$

Now write (7) as

$$v = r + K v$$

Finally, use the **Neumann series lemma**

$$r(K) < 1 \quad \implies \quad v = (I - K)^{-1} r$$

Interpretation

Recall that the condition is

$$r(K) < 1 \quad \text{where} \quad Kg(x) = \beta(x) \int g(y) q(x, y)$$

Gelfand's formula:

$$r(K) = \lim_{n \to \infty} \|K^n\|^{1/n}$$

Local spectral radius thm: If K is irreducible and eventually compact, then

$$r(K) = \lim_{n \to \infty} \|K^n g\|^{1/n} \quad \text{whenever } g \gg 0$$

Hence

$$r(K) = \lim_{n\to\infty} \|K^n \mathbb{1}\|^{1/n}$$

Since $Kg(x) = \beta(x) \int g(y)q(x,y)$, we have

$$(K^n \mathbb{1})(x) = \int \cdots \int \beta(x_0) \cdots \beta(x_{n-1}) q(x_0, x_1) \cdots q(x_{n-1}, x_{n-1})$$

$$= \mathbb{E}_x \prod_{t=0}^{n-1} \beta(X_t)$$

Thus,

$$r(K) = \lim_{n\to\infty} \left\| \mathbb{E}_x \prod_{t=0}^{n-1} \beta(X_t) \right\|^{1/n}$$

Now specialize to $\|\cdot\| = L_1(\pi)$ norm, so

$$\|f\| = \mathbb{E}|f(X_0)| \quad \text{when} \quad X_0 \sim \pi$$

Then

$$r(K) = \lim_{n\to\infty} \left\| \mathbb{E}_x \prod_{t=0}^{n-1} \beta(X_t) \right\|^{1/n}$$

$$= \lim_{n\to\infty} \left\{ \mathbb{E}\mathbb{E}_{X_0} \prod_{t=0}^{n-1} \beta(X_t) \right\}^{1/n}$$

$$= \lim_{n\to\infty} \left\{ \mathbb{E} \prod_{t=0}^{n-1} \beta(X_t) \right\}^{1/n} \approx \text{long run geometric average}$$

Example. If $\beta(X_t) \equiv \bar{\beta}$, then

$$\left\{ \mathbb{E} \prod_{t=0}^{n-1} \beta(X_t) \right\}^{1/n} = \left\{ \bar{\beta}^n \right\}^{1/n} = \bar{\beta}$$

Example. If $\{X_t\}$ is IID with $\bar{\beta} := \mathbb{E}\beta(X_t)$, then

$$\left\{ \mathbb{E} \prod_{t=0}^{n-1} \beta(X_t) \right\}^{1/n} = \left\{ \prod_{t=0}^{n-1} \mathbb{E}\beta(X_t) \right\}^{1/n} = \bar{\beta}$$

In either case,

$$r(K) < 1 \quad \iff \quad \bar{\beta} < 1$$

Example. Suppose $\{X_t\}$ obeys

$$X_{t+1} = \rho X_t + \mu + \sigma \eta_{t+1}, \qquad \{\eta_t\} \overset{\text{IID}}{\sim} N(0,1)$$

with $\rho \in (0,1)$ and $\beta(X_t) = \exp(X_t)$

Some algebra (see Stachurski and Zhang (2021)) gives

$$\lim_{n\to\infty} \left\{ \mathbb{E} \prod_{t=0}^{n-1} \beta(X_t) \right\}^{1/n} = \lim_{n\to\infty} \left\{ \mathbb{E} \exp\left(\sum_{t=0}^{n-1} X_t \right) \right\}^{1/n}$$

$$= \exp\left(\frac{\mu}{1-\rho} + \frac{\sigma^2}{2(1-\rho)^2} \right)$$

$$\therefore \quad r(K) < 1 \quad \iff \quad 2\mu + \frac{\sigma^2}{1-\rho} < 0$$

EZ Utility with Preference Shocks, Take 2

Epstein–Zin preferences defined recursively by

$$V_t = \left[(1-\beta)\lambda_t C_t^{1-1/\psi} + \beta \left\{\mathcal{R}_{t,1-\gamma}\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}$$

where

- $\mathcal{R}_{t,1-\gamma}$ is a **Kreps–Porteus** certainty equivalent operator with

$$\mathcal{R}_{t,1-\gamma}(V_{t+1}) = (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}$$

- $\{C_t\}_{t\geqslant 0}$ is a consumption path
- $\{\lambda_t\}_{t\geqslant 0}$ is a sequence of preference shocks
- V_t = utility value of $\{C_{t+j}\}_{j\geqslant 0}$

Consumption growth and the preference shock grow via

$$\ln\left(\frac{C_{t+1}}{C_t}\right) = g_c(X_t, X_{t+1}, \xi_{t+1})$$

and

$$\ln\left(\frac{\lambda_{t+1}}{\lambda_t}\right) = g_\lambda(X_t, X_{t+1}, \xi_{t+1})$$

where

- $\{X_t\}_{t \geqslant 0}$ is an aperiodic and irreducible Markov process on compact X
- $\{\xi_t\}_{t \geqslant 1}$ is IID on $\mathsf{Y} \subset \mathbb{R}^k$, and
- g_i is continuous for each $i \in \{c, \lambda\}$

Step 1. Let

$$G_t := \frac{1}{\lambda_t^\theta}\left(\frac{V_t}{C_t}\right)^{1-\gamma} \quad \text{with} \quad \theta := \frac{1-\gamma}{1-1/\psi}$$

Rewrite EZ recursion as

$$G_t = F\left[\mathbb{E}_t\, G_{t+1}\, \Gamma(X_t, X_{t+1}, \xi_{t+1})\right]$$

where

$$F(t) := \left(1 - \beta + \beta\, t^{1/\theta}\right)^\theta$$

$$\Gamma(x, y, z) := \exp\left\{\theta g_\lambda(x, y, z) + (1-\gamma) g_c(x, y, z)\right\}$$

Step 2 Convert

$$G_t = F\left[\mathbb{E}_t\, G_{t+1}\, \Gamma(X_t, X_{t+1}, \xi_{t+1})\right]$$

to

$$g(x) = F[(Kg)(x)]$$

where

$$(Kg)(x) = \mathbb{E}_x\, g(X_{t+1})\, \Gamma(X_t, X_{t+1}, \xi_{t+1})$$

Problem is now:

- solve for the fixed point g^* of $T := F \circ K$
- and obtain the solution $G_t^* = g^*(X_t)$

Summary Find the fixed point g^* of $T = F \circ K$ where

$$(Kg)(x) = \int g(y) \left[\int \Gamma(x, y, z) \nu(z) \, \mathrm{d}z \right] q(x, y) \, \mathrm{d}y$$

and

$$F(t) := \left(1 - \beta + \beta \, t^{1/\theta}\right)^{\theta}$$

Then set $G_t^* = g^*(X_t)$

Transform to get

- V_t = utility
- W_t = wealth-consumption ratio, etc.

But **which fixed point theorem to use**?

What about Banach's fixed point theorem?

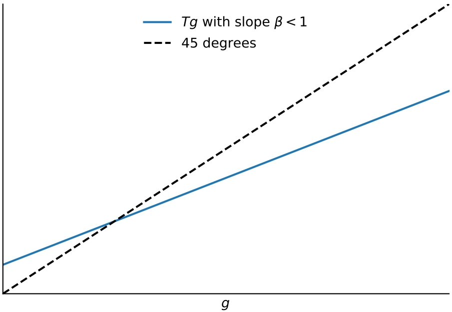


Figure: $|Tg - Th| \leqslant \beta |g - h|$

Consider the one-dimensional case

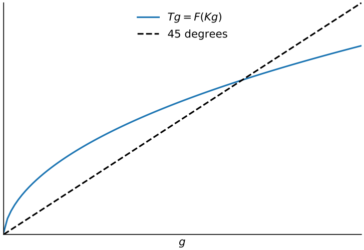


Figure: $Tg = F(Kg)$ when $g \in (0, \infty)$, $K = 1$, $\beta = 0.5$ and $\theta = -10$

Message: Banach will not work for all parameter values

The operator T is continuous and monotone

Should we use

- Brouwer's fixed point theorem?
- Schauder?
- Tarski?

What's the problem here?

We get our cue from this figure:

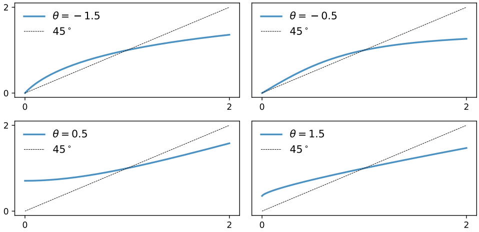


Figure: Shape properties of F

For any parameters, F is increasing and either convex or concave

$T = F \circ K$ and K is positive and linear, so true for T as well

Du's Theorem

The following theorem extends work by Yihong Du (1990)

Theorem Let $\mathscr{P}$ be the (nonempty) interior of the positive cone of a Banach lattice. Let $S : \mathscr{P} \to \mathscr{P}$ be order preserving and either convex or concave. Suppose further that, for any pair $g_1, g_2 \in \mathscr{P}$, there exists a pair $f_1, f_2 \in \mathscr{P}$ such that

1. $f_1 \leqslant g_1, g_2 \leqslant f_2$
2. $f_1 \ll S f_1$ and $S f_2 \ll f_2$

Then S has a unique fixed point g^* in $\mathscr{P}$ and, for all $g \in \mathscr{P}$,

$$\exists\, a < 1 \text{ such that } \|S^n g - g^*\| = \mathcal{O}(a^n)$$

Concave case

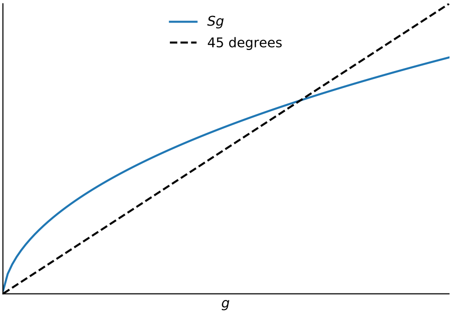


Figure: Concave and monotone increasing

Convex case

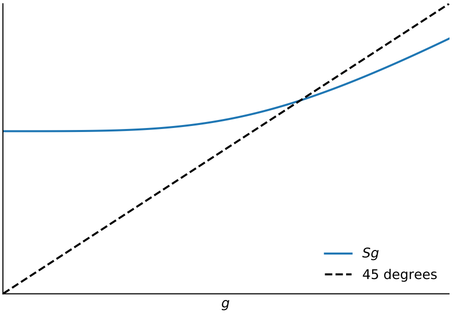


Figure: Convex and monotone increasing

Application to EZ Preference

The map $T = F \circ K$ is order preserving (increasing) and

- convex when $0 < \theta \leqslant 1$
- concave otherwise ($\theta = 0$ excluded)

Hence we need only check: $\forall\, g_1, g_2 \in \mathscr{P}$, $\exists\, f_1, f_2 \in \mathscr{P}$ such that

1. $f_1 \leqslant g_1, g_2 \leqslant f_2$
2. $f_1 \ll T f_1$ and $T f_2 \ll f_2$

Prop. This is true if and only if $\beta r(K)^{1/\theta} < 1$

Main Result

Let

$$\mathscr{S} := \ln \beta + \frac{1}{\theta} \ln(r(K))$$

Let $\mathscr{C}$ be the continuous functions from X to $(0, \infty)$

Theorem The following statements are equivalent:

(a) $\mathscr{S} < 0$

(b) T has a unique fixed point g^* in $\mathscr{C}$ and, for all $g \in \mathscr{C}$, there exists an $a < 1$ and $N < \infty$ such that

$$\|T^n g - g^*\|_\infty \leqslant a^n N \quad \text{for all } n \in \mathbb{N}$$

Moreover, if $\mathscr{S} \geqslant 0$, then no solution exists

Interpreting the Condition

Theorem If $\{C_t\}$ and $\{\lambda_t\}$ are independent, then

$$\mathcal{S} = \ln \beta + \mathcal{S}_\lambda + \left(1 - \frac{1}{\psi}\right) \mathcal{S}_c$$

where

$$\mathcal{S}_\lambda := \lim_{T \to \infty} \frac{1}{T} \ln \mathcal{R}_\theta \left(\frac{\lambda_T}{\lambda_0} \right)$$

$$\text{and} \quad \mathcal{S}_c := \lim_{T \to \infty} \frac{1}{T} \ln \mathcal{R}_{1-\gamma} \left(\frac{C_T}{C_0} \right)$$

Proof: Via a local spectral radius result by Krasnoselskii and Zima

Simple Example

Ignoring lack of compactness, suppose that

$$g_{\lambda,t+1} := \ln\left(\frac{\lambda_{t+1}}{\lambda_t}\right) = h_{\lambda,t+1}$$

where

$$h_{\lambda,t+1} = \rho_\lambda h_{\lambda,t} + s_\lambda \eta_{\lambda,t+1} \quad \text{and} \quad \{\eta_{\lambda,t+1}\} \overset{\text{IID}}{\sim} N(0,1),$$

Then

$$\mathcal{S}_\lambda = \theta \frac{s_\lambda^2}{2(1-\rho_\lambda)^2}$$

Key implication

$$\theta < 0 \quad \Longrightarrow \quad \mathcal{S}_\lambda < 0$$

Suppose further that (as in § I.A of Bansal and Yaron (2004))

$$g_{c,t+1} = \mu_c + z_t + \sigma_c \, \xi_{t+1}$$

$$z_{t+1} = \rho z_t + \sigma \, \eta_{t+1}$$

Then

$$\mathscr{S} = \ln \beta + \theta \, \frac{s_\lambda^2}{2(1-\rho_\lambda)^2} + \mu_c + \frac{1}{2}(1-\gamma)\left(\sigma_c^2 + \frac{\sigma^2}{(1-\rho)^2}\right)$$

Existence holds when

- patient
- risky preference shocks and consumption
- low mean growth rate for consumption

Testing the Condition for SSY

Let's look at Schorfheide, Song and Yaron (2018, ECMA)

Pref shocks are as above but

$$g_{c,t+1} := \ln\left(\frac{C_{t+1}}{C_t}\right) = \mu_c + z_t + \sigma_{c,t}\,\xi_{c,t+1},$$

where

$$z_{t+1} = \rho\, z_t + \sigma_{z,t}\,\eta_{t+1}$$

and

$$\sigma_{i,t} = \phi_i \, \exp(h_{i,t})$$

$$h_{i,t+1} = \rho_i\, h_{i,t} + s_i\, \eta_{i,t+1} \quad \text{for } i \in \{z, c\}$$

No analytical solution for $\mathcal{S}_c$ exists

But recall that

$$\mathcal{S} = \ln \beta + \frac{1}{\theta} \ln(r(K))$$

After discretization,

$$K(x, y) = \mathbb{E} \, \exp\left\{\theta g_\lambda(x, y, \xi) + (1 - \gamma) g_c(x, y, \xi)\right\} q(x, y)$$

$$q(x, y) = \text{ discretized state dynamics}$$

Hence

- Compute the matrix K
- Compute dominant eigenvalue (which $= r(K)$)

Let RAR1(ρ, σ) := **Rouwenhorst discretization** of a centered Gaussian AR1 with params σ, ρ

$h_\lambda[\ell], P_\lambda[\ell, :]$ for $\ell = 1, \ldots L \ \leftarrow$ RAR1$(\rho_\lambda, s_\lambda)$

$h_c[k], P_c[k, :]$ for $k = 1, \ldots K \ \leftarrow$ RAR1(ρ_c, s_c)

$h_z[i], P_z[i, :]$ for $i = 1, \ldots I \ \leftarrow$ RAR1(ρ_z, s_z)

for $i \in \{1, \ldots, I\}$ **do**

  $\sigma_z[i] \leftarrow \phi_z \exp(h_z[i])$

  $z[i, j], Q_z[i, j, :]$ for $j = 1, \ldots J \ \leftarrow$ RAR1$(\rho, \sigma_z[i])$

end

Now map the multi-index to a single index:

$$m = \ell(K \cdot I \cdot J) + k(I \cdot J) + iJ + j$$

$$M = L \cdot K \cdot I \cdot J$$

for m *in* $1, \ldots, M$ **do**
    get (ℓ, k, i, j) from m
    $x[m] \leftarrow (h_\lambda[\ell], h_c[k], h_z[i], z[i,j])$
    for m' *in* $1, \ldots, M$ **do**
        get (ℓ', k', i', j') from m'
        $q[m, m'] \leftarrow P_\lambda[\ell, \ell'] P_c[k, k'] P_z[i, i'] Q_z[i, j, j']$
    end
end

Now compute the $M \times M$ matrix K and set

$$\mathcal{S} = \ln \beta + \frac{1}{\theta} \ln(r(K))$$

Let d = number of states for each Rouwenhorst discretization

Then $M = L \cdot K \cdot I \cdot J = d^4$

Example.

- $d = 6 \implies M = 1296$
- $d = 12 \implies M = 20736$

Compute $r(K)$ using QR algorithm in LAPACK

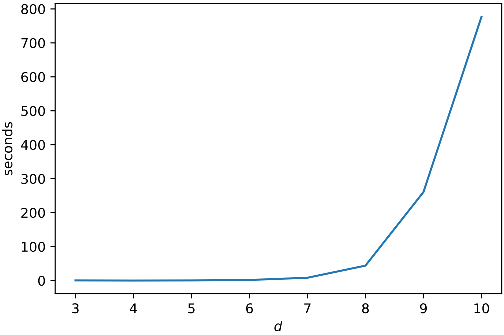


Figure: Compute time as a function of d

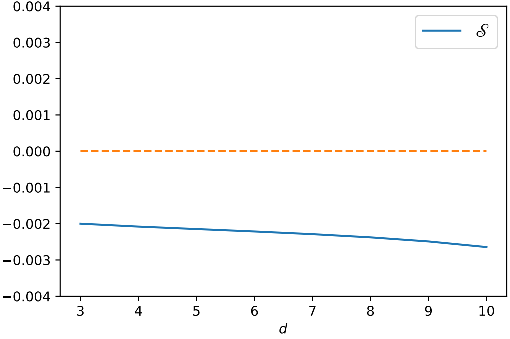


Figure: SSY stability coefficient $\mathscr{S}$ as a function of d

Only issue is the compute time

- Finer discretizations are closer to the original

And what happens if we add two more state variables?

Example. If $M = H \cdot G \cdot L \cdot K \cdot I \cdot J = d^6$, then

- $d = 6 \implies M = 46,656$
- $d = 12 \implies M = 2,985,984$

Memory requirement when $d = 12$ for 64 bit floats:

$$71,328,803,586,048 \text{ bytes } = 71,328 \text{ GB}$$

GPU-Based Alternative

Recall

$$\mathcal{S} = \ln\beta + \mathcal{S}_\lambda + \left(1 - \frac{1}{\psi}\right)\mathcal{S}_c$$

with

$$\text{and} \quad \mathcal{S}_c := \lim_{T\to\infty} \frac{1}{T} \ln \mathcal{R}_{1-\gamma}\left(\frac{C_T}{C_0}\right)$$

Approximate via Monte Carlo

$$\mathcal{R}_{1-\gamma}\left(\frac{C_T}{C_0}\right) = \left[\mathbb{E}\left(\frac{C_T}{C_0}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \approx \left[\frac{1}{N}\sum_{n=1}^{N}\left(\frac{C_T^{(n)}}{C_0^{(n)}}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

GPU evaluation of

$$\left[\frac{1}{N} \sum_{n=1}^{N} \left(\frac{C_T^{(n)}}{C_0^{(n)}} \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$$

do in parallel

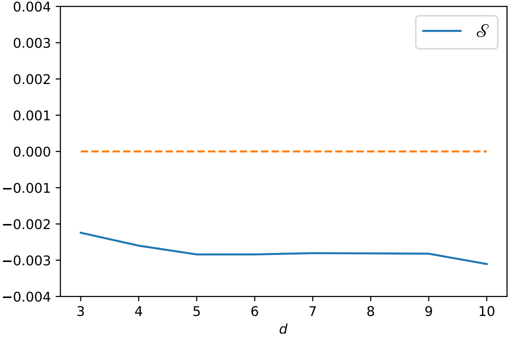
$$a_1 \leftarrow \left(C_T^{(1)} / C_0^{(1)} \right)^{1-\gamma}$$

$$\vdots$$

$$a_N \leftarrow \left(C_T^{(N)} / C_0^{(N)} \right)^{1-\gamma}$$

end

return $\left[(1/N) \sum_{n=1}^{N} a_n \right]^{\frac{1}{1-\gamma}}$



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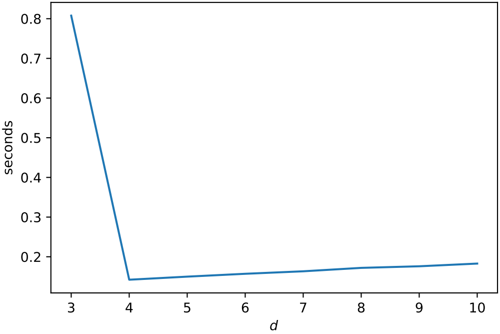

Figure: Compute time as a function of d

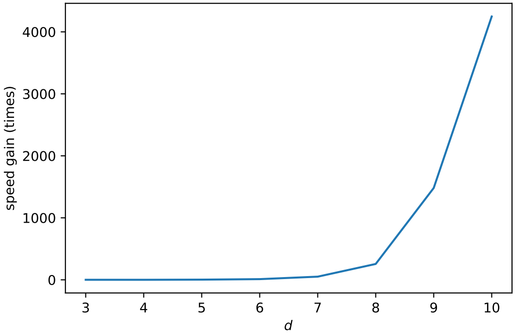

Figure: Relative compute time (CPU/GPU) as a function of d

Computing Recursive Utility

Now we know that

- $\exists$ a unique $g^* \in \mathscr{C}$ such that $g^* = Tg^*$
- $T^n g \to g^*$ as $n \to \infty$ for all $g \in \mathscr{C}$

From this we can obtain recursive utility

Method: fix $g \in \mathscr{C}$ and iterate on

$$Tg = \left(1 - \beta + \beta (Kg)^{1/\theta}\right)^{\theta}$$

Computing the WC Ratio

To compute $\{M_t\}$, we need the **wealth-consumption ratio**, which is the fixed point of

$$Uw = (1 + \beta K w^{\theta})^{1/\theta}$$

Proposition The following statements are equivalent:

1. $\mathscr{S} < 0$
2. U has a unique and globally stable fixed point w^* in $\mathscr{C}$

Proof: Let $\tau : \mathscr{C} \to \mathscr{C}$ be defined by

$$\tau g = \frac{1}{1 - \beta} g^{1/\theta}$$

Then $U = \tau T \tau^{-1}$ on $\mathscr{C}$

Visualization of $U = \tau T \tau^{-1}$ on $\mathscr{C}$:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{T} & \mathscr{C} \\ {\scriptstyle \tau^{-1}}\downarrow & & \uparrow{\scriptstyle \tau} \\ \mathscr{C} & \xrightarrow{U} & \mathscr{C} \end{array}$$

$\therefore \quad (\mathscr{C}, T)$ and $(\mathscr{C}, U)$ are topologically conjugate

$\therefore \quad (\mathscr{C}, U)$ is globally stable $\iff$ $(\mathscr{C}, T)$ is globally stable

$\therefore \quad (\mathscr{C}, U)$ is globally stable $\iff \quad \mathscr{S} < 0$

Hence we compute $w^* = Uw^*$ by successive approximation

- Fix $w \in \mathscr{C}$
- Iterate on $Uw = (1 + \beta K w^\theta)^{1/\theta}$

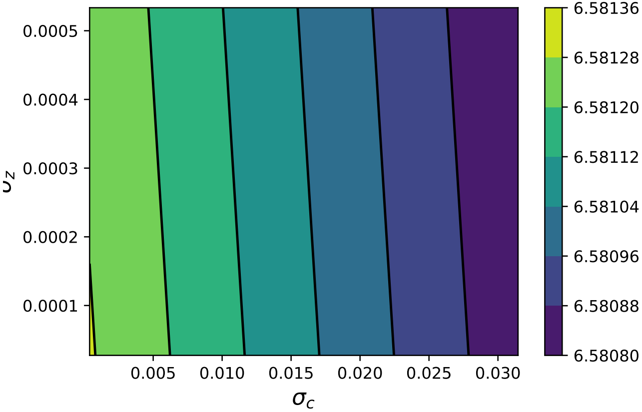

Figure: WC ratio when $d = 10$ with z and h_λ fixed

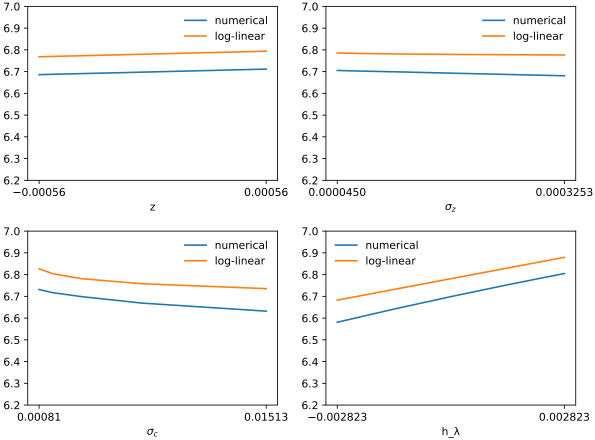

Figure: WC ratio when $d = 5$

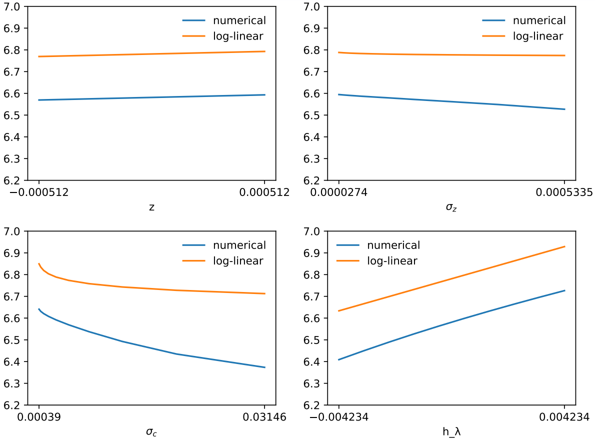

Figure: WC ratio when $d = 10$

Parallelized Iteration on the GPU

Fix initial g

do

  do in parallel

    Compute $Kg(x_1)$

    $\vdots$

    Compute $Kg(x_M)$

  end

  $Tg \leftarrow \left(1 - \beta + \beta\,(Kg)^{1/\theta}\right)^{\theta}$

  $\epsilon \leftarrow \|Tg - g\|_{\infty}$

  $g \leftarrow Tg$

while $\epsilon > tol$

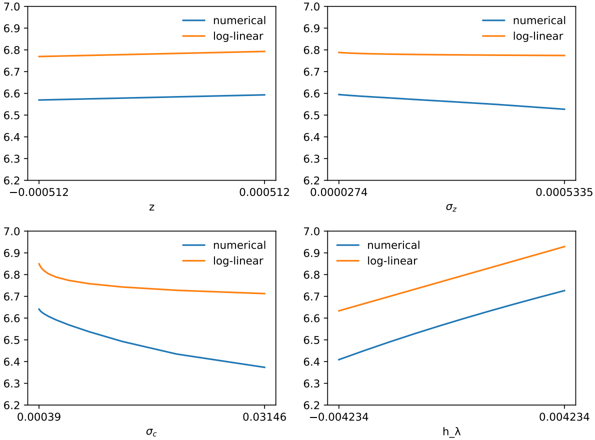

Figure: GPU based computation of WC ratio when $d = 10$

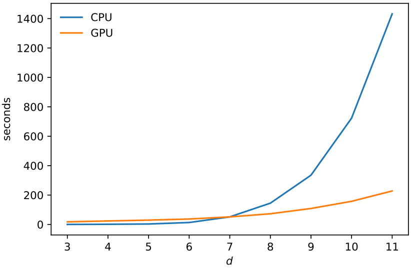

Figure: GPU time (manual parallelization) vs CPU time

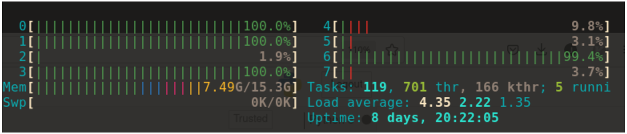

Figure: Matrix WC computations on the CPU

CuPy Implementation

```
# Transfer arrays to the GPU
K = cp.asarray(K_matrix)
w = cp.asarray(w)

while error > tol and iter < max_iter:
    Tw = 1 + beta * (cpm(K, (w**theta)))**(1/theta)
    error = cp.max(cp.abs(w - Tw))
    w = Tw
    iter += 1

# Transfer back to the host
w = cp.asnumpy(w)
```

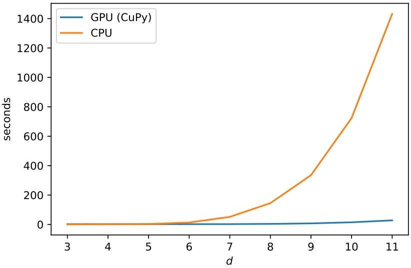

Figure: GPU time (CuPy implementation) vs CPU time

Next Steps

- Calculate $\{M_t\}$
- Calculate prices and returns given $\{M_t\}$
- Repeat for Gomez-Cram and Yaron (2020) (6 states)