Advanced Econometric Methods EMET3011/8014

Lecture 12

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Announcements/Reminders

- Please get a fresh copy of the course notes PDF
- No more changes to course notes
- No computer lab today
- Today's lab time is extra office hours
- After today you can meet me/Varang by appointment

Today's Lecture

- Large Sample OLS

Large Sample OLS

We now return to the linear regression problem

Introduce more reasonable assumptions

Establish consistency and asymptotic normality

Assumptions

We retain the linear model assumption

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t \qquad (t = 1, \ldots, T)$$

- Notation: t (resp., T) instead of n (resp., N)

The other OLS assumptions are adjusted as follows:

Assumption. (Ergodic Regressors): The sequence $\{\mathbf{x}_t\}$ is

1. identically distributed
2. satisfies $\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t' \xrightarrow{p} \mathbb{E}\left[\mathbf{x}_t\mathbf{x}_t'\right] =: \boldsymbol{\Sigma}_{\mathbf{xx}}$ as $T \to \infty$
3. $\boldsymbol{\Sigma}_{\mathbf{xx}}$ is positive definite

Note: Independence is not required

Example: Markov process with

- Unique, globally stable stationary distribution π_∞
- $\mathbf{x}_1 \sim \pi_\infty$
- $\mathbb{E}\left[\mathbf{x}_t\mathbf{x}_t'\right]$ positive definite

Assumption. (Weak Exogeneity):

1. $\{u_t\}$ is IID with $\mathbb{E}[u_t] = 0$ and $\mathbb{E}[u_t^2] = \sigma^2$
2. u_t and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ independent for all t

Note: Current shocks permitted to affect future regressors

Example: AR(1) model with IID shocks (see last lecture)

- $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t =$ lagged state variables $y_0, \ldots, y_{t-1}$
- These are functions of $u_0, \ldots, u_{t-1}$
- Hence independent of u_t

Estimators

Expression for estimator of $\boldsymbol{\beta}$ is unchanged:

$$\hat{\boldsymbol{\beta}}_T := (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

We require as before that $\mathbf{X}$ is full column rank

To estimate σ^2 we use

$$\hat{\sigma}_T^2 := \frac{\mathrm{SSR}}{T}$$

(Not $T - K$ because the difference will be insignificant)

Recall that

$$\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$$

Multiplying and dividing by T,

$$\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} = \left[\frac{1}{T}\mathbf{X}'\mathbf{X}\right]^{-1} \cdot \ \frac{1}{T}\mathbf{X}'\mathbf{u}$$

Expanding out the matrix products (exercise), we obtain

$$\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} = \left[\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]^{-1} \cdot \ \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t u_t$$

What happens to these two sample means as $T \to \infty$?

Lemma. If ER and WE hold, then

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t u_t \xrightarrow{p} \mathbf{0}$$

and

$$T^{-1/2}\sum_{t=1}^{T}\mathbf{x}_t u_t \xrightarrow{d} \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \sigma^2\boldsymbol{\Sigma}_{\mathbf{xx}})$$

Proof: Suffices to show that, for any vector $\mathbf{a} \in \mathbb{R}^K$,

$$\mathbf{a}'\left[\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t u_t\right] \xrightarrow{p} \mathbf{a}'\mathbf{0} \quad \text{and} \quad \mathbf{a}'\left[T^{-1/2}\sum_{t=1}^{T}\mathbf{x}_t u_t\right] \xrightarrow{d} \mathbf{a}'\mathbf{z}$$

Equivalent: Suffices to show that

$$\frac{1}{T}\sum_{t=1}^{T} m_t \xrightarrow{p} 0 \quad \text{and} \quad T^{-1/2}\sum_{t=1}^{T} m_t \xrightarrow{d} \mathcal{N}(0, \sigma^2 \mathbf{a}'\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{a})$$

where

$$m_t := \mathbf{a}'\mathbf{x}_t u_t$$

Strategy:

- Show that $\{m_t\}$ is an MDS
- Apply MDS LLN/CLT

Let $\mathcal{F}_t := \{\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{x}_{t+1}, u_1, \ldots, u_t\}$

Claim: If ER and WE hold, then $m_t := \mathbf{a}'\mathbf{x}_t u_t$ is

1. a MDS with respect to $\{\mathcal{F}_t\}$
2. identically distributed with $\mathbb{E}\left[m_t^2\right] = \sigma^2 \mathbf{a}' \mathbf{\Sigma}_{\mathbf{xx}} \mathbf{a}$
3. satisfies $\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left[m_t^2 \mid \mathcal{F}_{t-1}\right] \xrightarrow{p} \sigma^2 \mathbf{a}' \mathbf{\Sigma}_{\mathbf{xx}} \mathbf{a}$

The MDS LLN/CLT now yields claim on previous slide:

$$\frac{1}{T}\sum_{t=1}^{T} m_t \xrightarrow{p} 0 \qquad \text{and} \qquad T^{-1/2}\sum_{t=1}^{T} m_t \xrightarrow{d} \mathcal{N}(0, \sigma^2 \mathbf{a}' \mathbf{\Sigma}_{\mathbf{xx}} \mathbf{a})$$

▷ Proof that $m_t = \mathbf{a}'\mathbf{x}_t u_t$ is a MDS with respect to

$$\mathcal{F}_t := \{\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{x}_{t+1}, u_1, \ldots, u_t\}$$

Clearly $\{m_t\}$ is adapted to $\{\mathcal{F}_t\}$

Moreover,

$$\begin{aligned}
\mathbb{E}\left[m_{t+1} \,|\, \mathcal{F}_t\right] &= \mathbb{E}\left[u_{t+1}\mathbf{a}'\mathbf{x}_{t+1} \,|\, \mathcal{F}_t\right] \\
&= \mathbf{a}'\mathbf{x}_{t+1}\mathbb{E}\left[u_{t+1} \,|\, \mathcal{F}_t\right] \qquad (\because\ \mathbf{x}_{t+1} \in \mathcal{F}_t) \\
&= \mathbf{a}'\mathbf{x}_{t+1}\mathbb{E}\left[u_{t+1}\right] \\
&= \mathbf{a}'\mathbf{x}_{t+1}\, 0 \\
&= 0
\end{aligned}$$

▷ Proof that $\mathbb{E}\left[m_t^2\right] = \sigma^2 \mathbf{a}' \boldsymbol{\Sigma}_{\mathbf{xx}} \mathbf{a}$

$$\mathbb{E}\left[m_t^2\right] = \mathbb{E}\left[u_t^2 (\mathbf{a}'\mathbf{x}_t)^2\right] = \mathbb{E}\left[u_t^2\right] \mathbb{E}\left[(\mathbf{a}'\mathbf{x}_t)^2\right] = \sigma^2 \mathbb{E}\left[(\mathbf{a}'\mathbf{x}_t)^2\right]$$

$$(\mathbf{a}'\mathbf{x}_t)^2 = \mathbf{a}'\mathbf{x}_t\mathbf{a}'\mathbf{x}_t = \mathbf{a}'\mathbf{x}_t\mathbf{x}_t'\mathbf{a} \text{ by symmetry of inner product}$$

$$\therefore \quad \mathbb{E}\left[(\mathbf{a}'\mathbf{x}_t)^2\right] = \mathbf{a}'\mathbb{E}\left[\mathbf{x}_t\mathbf{x}_t'\right]\mathbf{a} = \mathbf{a}'\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{a}$$

$$\therefore \quad \mathbb{E}\left[m_t^2\right] = \sigma^2 \mathbf{a}'\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{a}$$

▷ Proof that $\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[m_t^2 \mid \mathcal{F}_{t-1}\right] \xrightarrow{p} \sigma^2\mathbf{a}'\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{a}$

Since $\mathbf{x}_t \in \mathcal{F}_{t-1}$, we have

$$\mathbb{E}\left[m_t^2 \mid \mathcal{F}_{t-1}\right] = \mathbb{E}\left[u_t^2(\mathbf{a}'\mathbf{x}_t)^2 \mid \mathcal{F}_{t-1}\right] = \sigma^2(\mathbf{a}'\mathbf{x}_t)^2 = \sigma^2\mathbf{a}'\mathbf{x}_t\mathbf{x}_t'\mathbf{a}$$

$$\therefore \quad \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[m_t^2 \mid \mathcal{F}_{t-1}\right] = \frac{1}{T}\sum_{t=1}^{T}(\sigma^2\mathbf{a}'\mathbf{x}_t\mathbf{x}_t'\mathbf{a}) = \sigma^2\mathbf{a}'\left[\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]\mathbf{a}$$

Applying ER, we get $\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t' \xrightarrow{p} \mathbf{\Sigma}_{\mathbf{xx}}$

$$\therefore \quad \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[m_t^2 \mid \mathcal{F}_{t-1}\right] \xrightarrow{p} \sigma^2\mathbf{a}'\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{a}$$

That was a lot of work...

But we can now easily prove:

- Consistency of $\hat{\boldsymbol{\beta}}_T$
- Consistency of $\hat{\sigma}_T^2$
- Asymptotic normality of $\hat{\boldsymbol{\beta}}_T$

In the next three slides we assume ER and WE hold

Theorem. $\hat{\boldsymbol{\beta}}_T \xrightarrow{p} \boldsymbol{\beta}$ as $T \to \infty$

Proof: Recall that

$$\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} = \left[\frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right]^{-1} \cdot \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t u_t$$

$$\left[\frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right]^{-1} \xrightarrow{p} \boldsymbol{\Sigma}_{\mathbf{xx}}{}^{-1} \qquad \text{and} \qquad \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t u_t \xrightarrow{p} \mathbf{0}$$

$$\therefore \quad \hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} \xrightarrow{p} \boldsymbol{\Sigma}_{\mathbf{xx}}{}^{-1} \mathbf{0} = \mathbf{0}$$

$$\therefore \quad \hat{\boldsymbol{\beta}}_T \xrightarrow{p} \boldsymbol{\beta}$$

Theorem. $\hat{\sigma}_T^2 := \frac{\text{SSR}}{T} \xrightarrow{p} \sigma^2$

Proof: Using $y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t$ and $\boldsymbol{\gamma}_T := \boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_T$, we have

$$\hat{\sigma}_T^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}}_T)^2 = \frac{1}{T}\sum_{t=1}^{T}(u_t + \mathbf{x}_t'\boldsymbol{\gamma}_T)^2$$

Expanding out the square,

$$\begin{aligned}\hat{\sigma}_T^2 &= \frac{1}{T}\sum_{t=1}^{T} u_t^2 + 2\gamma_T'\left[\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t u_t\right] + \gamma_T'\left[\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right]\gamma_T \\ &\xrightarrow{p} \sigma^2 + 0 + 0\end{aligned}$$

- Check details
- Check same true if $\hat{\sigma}_T^2 := \frac{\text{SSR}}{T-K}$

Asymptotic Normality

Theorem. $\sqrt{T}(\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1})$

Proof: Observe that

$$\sqrt{T}(\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}) = \left[\frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]^{-1} \cdot \left[T^{-1/2}\sum_{t=1}^{T} \mathbf{x}_t u_t\right]$$

$$T^{-1/2}\sum_{t=1}^{T} u_t \mathbf{x}_t \xrightarrow{d} \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{\Sigma}_{\mathbf{xx}})$$

$$\therefore \quad \sqrt{T}(\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}) \xrightarrow{d} \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\operatorname{var}[\mathbf{z}]\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1})$$

Result follows because $\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\operatorname{var}[\mathbf{z}]\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1} = \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\sigma^2\boldsymbol{\Sigma}_{\mathbf{xx}}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1} = \sigma^2\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}$

Finally:

- Keep checking course home page — practice questions coming
- Review the lecture slides — check the details!

Exam:

- Will cover the whole course with emph on latter half
- No R questions, just analytical exercises