Interest Rate Dynamics and Commodity Prices[†]

Christophe Gouel[◇], Qingyin Ma[*], and John Stachurski[⋆]

[◇]Université Paris-Saclay, INRAE, AgroParisTech, Paris-Saclay Applied Economics, and CEPII, 20 avenue de Ségur, Paris, France

[*]ISEM, Capital University of Economics and Business

[⋆]Research School of Economics, Australian National University

September 7, 2024

ABSTRACT. In economic studies and popular media, interest rates are routinely cited as a major factor behind commodity price fluctuations. At the same time, the channels of transmission are far from transparent, leading to long-running debates on the sign and magnitude of interest rate effects. Purely empirical studies struggle to address these issues because of the complex interactions between interest rates, prices, supply changes and aggregate demand. To move this debate to a solid footing, we extend the competitive storage model to include stochastically evolving interest rates. We establish general conditions for existence and uniqueness of solutions, as well as providing a systematic theoretical and quantitative analysis of the interactions between interest rates and prices.

Keywords: commodity prices, time-varying interest rate, competitive storage.

JEL Classification: C62, C63, E43, E52, G12, Q02.

1. INTRODUCTION

Commodity prices are major determinants of exchange rates, government revenue, the balance of payments, output fluctuations, and inflation (see, e.g., Byrne et al., 2013; Gospodinov and Ng, 2013; Eberhardt and Presbitero, 2021; Peersman, 2022). While some commodity price movements are driven by idiosyncratic shocks, Alquist et al. (2020) find that up to 80% of the variance of commodity prices is explained by common factors (see also Byrne et al., 2013). Aggregate factors are particularly important when considering the impact of commodities on inflation and exchange rates because such factors induce price comovement in all or many commodities.

Historically, the aggregate factor that has generated the most attention is interest rates. For example, Frankel (2008b, 2008c, 2018) has long argued that interest rates are a major driver of comovements in commodity prices, with rising interest rates decreasing commodity prices and falling interest rates increasing them. The main argument relates to

[†]We thank Erica Perego, John Rust, Liyan Yang, Changhua Yu, Shenghao Zhu, and seminar audiences at CEPII, CUHK-SZ, Peking University, and XJTU for very helpful comments and suggestions. Qingyin Ma gratefully acknowledges the financial support from Natural Science Foundation of China (No. 72003138) and the Project of Construction and Support for High-Level Innovative Teams of Beijing Municipal Institutions (No. BPHR20220119).

E-mail addresses: christophe.gouel@inrae.fr, qingyin.ma@cueb.edu.cn, and john.stachurski@anu.edu.au

cost of carry: higher interest rates reduce demand for inventories, which exerts downward pressure on commodity prices. At the same time, it is easy to imagine scenarios where interest rates and commodity prices are *positively* correlated—for example, when high aggregate demand boosts both commodity prices and the cost of borrowing (through credit markets and, potentially, the responses of monetary authorities).

Indeed, empirical studies of the sign and magnitude of interest rate effects on commodity prices face deep challenges because of the endogeneity and equilibrium nature of the mechanisms in question. For example, even if we fully control for changes in output and demand, rising commodity prices might themselves trigger a tightening of monetary policy, without any change in output (see, e.g., Cody and Mills, 1991). Conversely, pure monetary shocks affect commodity markets through various channels (e.g., speculation, aggregate demand, and supply response) that are hard to disentangle empirically.[1]

These challenges demand a structural model built on firm theoretical foundations that can isolate the direct effect of interest rates on commodity prices through each of the channels listed above. The obvious candidate to provide the necessary structure is the competitive storage model developed by Samuelson (1971), Newbery and Stiglitz (1982), Wright and Williams (1982), Scheinkman and Schechtman (1983), Deaton and Laroque (1992, 1996), Chambers and Bailey (1996) and Cafiero et al. (2015), among others. In this model, commodities are assets that also have intrinsic value, separate from future cash flows. The standard version of the model features time-varying production, storage by forward-looking investors, arbitrage constraints, and non-negative carryover. Within the constraints of the model, there is a clear relationship between interest rates, storage, and commodity prices. Fama and French (1987) show that the relationship between the basis, the spread between the futures and spot prices, and interest rates is consistent with the structure of this model.

The main obstacle to applying the standard competitive storage model to the problem at hand is that the discount rate is constant. The source of this shortcoming is technical: a constant positive interest rate is central to the traditional proof of the existence and uniqueness of equilibrium prices and the study of their properties (see, e.g., Deaton and Laroque, 1992, 1996). In particular, positive constant rates are used to obtain contraction mappings over a space of candidate price functions, with the discount factor being the modulus of contraction.

At the same time, relaxing the assumption of constant discounting is necessary for analysis of interactions between interest rates and commodity prices. Without this modification, it is not possible to study how the nature and timing of shocks to supply, demand, and interest rates affect the sign and magnitude of changes in commodity prices. Moreover, allowing for state-dependent discounting brings the model closer to the data, since real interest rates do exhibit large movements over time, as shown in Figure 1.

[1]For example, a decline in the US interest rate can stimulate both global demand (see, e.g., Ramey, 2016) and firms' incentive to hold inventories (see, e.g., Frankel, 1986, 2008a, 2014), which then increase commodity prices. An increase in interest rates works in the opposite direction.

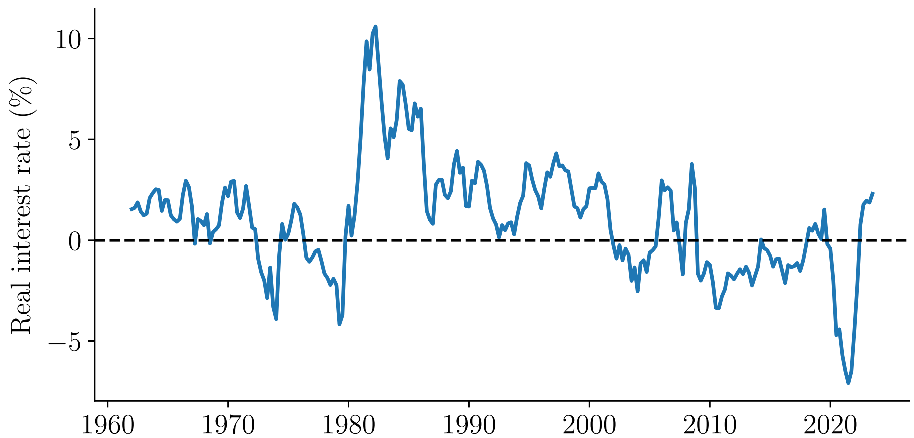

FIGURE 1. The real interest rate over the long run (one-year US treasury yield deflated by a measure of expected inflation obtained from an autoregressive model estimated on a 30-year rolling window). Source: FRED.

One difficulty with relaxing the assumption of constant interest rates in the competitive storage model is that negative real interest rates cannot be ignored, as is clear from Figure 1. If interest rates can be sufficiently negative for sufficiently long periods, then the model will have no finite equilibrium, due to unbounded demand for inventories. Thus, developing a model that can handle realistic calibrations requires accommodating negative yields on risk-free bonds in some states of the world, while providing conditions on these states and the size of the yields such that the model retains a well-defined and unique solution.

In this paper, we extend the competitive storage model to include state-dependent discounting and establish conditions under which a unique equilibrium price process exists. These conditions allow for both positive and negative discount rates, while also providing a link between the asymptotic return on risk-free assets with long maturity, the depreciation rate of the commodity in question, and the existence and uniqueness of solutions. Under these conditions, we show that the equilibrium solution can be computed via a globally convergent algorithm and provide a characterization of the continuity and monotonicity properties of the equilibrium objects. We also develop an endogenous grid algorithm for computing equilibrium objects efficiently.

With these results in hand, we examine the effect of interest rates on commodity prices from a theoretical and quantitative perspective. We show that, in some settings, interest rates and commodity prices can be positively correlated, such as when shocks shift up both interest rates and aggregate demand. Nevertheless, we are able to identify relatively sharp conditions under which a negative correlation is realized and analyze the impact of interest rates on commodity prices in depth. These conditions require that the exogenous state follows a monotone Markov process that is independent across dimensions and has a non-negative effect on interest rate and commodity output. The independence restriction cannot be dropped: if different exogenous states are contemporaneously correlated, then the relationship between interest rates and commodity prices can be reversed.

On the quantitative side, we study the impulse response functions (IRFs) of commodity price, inventory, and price volatility in response to an interest rate shock for suitable structural parameters. We then use these IRFs to explore the speculative and global demand channels. The former examines the role of speculators in the physical market, whose incentives to hold inventories are affected by fluctuations in interest rates. The latter studies the impact of exogenous interest rate shocks on commodity demand through their effects on economic activity. While both channels have been frequently suggested in the literature, their analysis remains significantly under-explored to date. To capture the nonlinear dynamics of the competitive storage model, we follow the methodology of Koop et al. (1996), wherein IRFs are defined as state-and-history-dependent random variables.

The simulated IRFs show that prices fall immediately after a positive interest rate shock and slowly converge to their long-run value, with a more pronounced decline and a slower convergence when the demand channel is active. The behavior of inventory dynamics is nuanced. While inventories typically decrease post-shock due to higher cost of carry, they may later rise when the demand channel is active. This is because reduced demand lowers spot prices, creating profit opportunities for storage. Overall, inventories tend to return to their long-run average more slowly than prices. Furthermore, price volatility exhibits sensitivity to inventory dynamics: a larger response in inventory tends to generate an inversely larger response in price volatility. Finally, the magnitude and overall pattern of the IRFs depend substantially on the market supply and interest rate regimes.[2]

Our work has some implications concerning commodity futures. For example, following the methodology of Gardner (1976), it has become common to use futures prices as a proxy for expected prices (see, e.g., Gouel and Legrand, 2022). However, our findings indicate that this substitution is invalid within the framework outlined in this study. As elucidated by Cox et al. (1981), forward and futures prices diverge in the presence of stochastic interest rates. Considering the potential for a strong correlation between interest rates and commodity prices—particularly under the global demand channel—stochastic interest rates are likely to affect the costs associated with margin requirements in futures markets. This, in turn, can lead to distinct behaviors in forward and futures price dynamics.

Regarding existing literature on interest rates and commodity prices, Jeffrey Frankel has made numerous empirical and theoretical contributions to this topic, focusing on how commodity prices overshoot their long-run target after a shock due to their inherent price flexibility (Frankel and Hardouvelis, 1985; Frankel, 1986, 2008a, 2014). This literature tends to find a negative effect of interest rate increases on commodity prices in the short

[2]These findings suggest that postulating a uniform effect of monetary shocks across different market supply and interest rate scenarios could bias empirical analysis.

(Rosa, 2014; Scrimgeour, 2015) and medium run (Anzuini et al., 2013; Harvey et al., 2017).[3] A negative relationship between interest rates and commodity prices has also been found by Christiano et al. (1999) and Bernanke et al. (2005). Moreover, interest rates affect not only the level of commodity prices, but also their cross-correlation and their volatility (Gruber and Vigfusson, 2018). Compared to these studies, the methodology developed here allows for a more systematic analysis of transmission mechanisms, clarifying the respective role of speculative and demand channels.

This work also intersects with other studies that examine the theoretical relationship between interest rate fluctuations and commodity prices, including Arseneau and Leduc (2013), Basak and Pavlova (2016), Tumen et al. (2016), and Bodart et al. (2021). However, Basak and Pavlova (2016) omit the nonnegativity constraint on storage, while Tumen et al. (2016) and Bodart et al. (2021) overlook the nonlinearity of storage and impact of large shocks by approximating their model around a steady state with positive stocks. Although Arseneau and Leduc (2013) admit nonlinearity, they adopt a limited stochastic structure where commodity production shocks are the only source of uncertainty. In contrast to these studies, we establish a comprehensive theory that avoids these simplifications and, in addition, allows us to jointly handle realistic depreciation rates and interest rate processes, which are crucial for accurately representing commodity price dynamics.[4] Moreover, we furnish a general theory on the existence and uniqueness of equilibrium solutions in this framework, with our assumptions being almost necessary.[5]

On the empirical side, a large literature has analyzed the empirical validity of the storage model (e.g., Deaton and Laroque, 1996; Cafiero et al., 2011, 2015; Gouel and Legrand, 2022). This literature focuses on idiosyncratic shocks and neglects shocks to storage costs. In contrast, we study the role of aggregate shocks on storage costs, providing a theoretical analysis of conditions under which interest rates have a negative effect on commodity prices and a quantitative analysis on the impact of interest rate shocks through speculative and global demand channels.

From a technical perspective, our work has some overlap with recent work on household and consumption problems with state-dependent discounting. For example, Ma et

[3]An exception is Kilian and Zhou (2022), who find no effect of real interest rate movements on oil prices.

[4]Our framework admits occasionally negative interest rates, in line with real-world outcomes. While some earlier models also allow negative interest rates, the relevant parameterizations are not empirically plausible, since, in these models, negative interest rates must be offset by excessively large depreciation rates in order to obtain an equilibrium.

[5]While stochastic interest rates are relatively novel in the storage model framework, work in the finance literature has shown that commodity pricing models benefit from incorporating stochastic convenience yields or stochastic interest rates (see, e.g., Gibson and Schwartz, 1990; Schwartz, 1997; Casassus and Collin-Dufresne, 2005). This literature focuses on questions somewhat orthogonal to our interests, such as term structure. One bridge between the approaches is the study by Routledge et al. (2000), which suggests that within a storage framework akin to the one we are exploring, convenience yields naturally arise from the interaction of supply, demand, and storage dynamics. We also note recent studies that examine commodity financialization and speculation through financial derivatives, including works by Basak and Pavlova (2016), Baker (2021), Goldstein and Yang (2022), although we do not pursue these topics in our paper.

al. (2020) use Euler equation methods to obtain existence and uniqueness of solutions to an optimal savings problem in a setting where the subjective discount rate is state-dependent. Stachurski and Zhang (2021) and Toda (2021) study similar problems. Like those papers, we tie stochastic discounting to long-run "eventual" contraction methods. Unlike those papers, we apply eventual contraction methods to commodity pricing problems.

The rest of the paper is organized as follows. Section 2 formulates a rational expectations competitive storage model with time-varying discounting and discusses the existence, uniqueness, and computability of the equilibrium solutions. Sections 3 and 4 examine the role of interest rates on commodity prices from a theoretical and quantitative perspective, respectively. Section 5 concludes. Proofs, descriptions of algorithms, and counterexamples can be found in the appendices.

2. Equilibrium Prices

This section formulates the competitive storage model with time-varying discounting and discusses conditions under which existence and uniqueness of the equilibrium pricing rule hold.

2.1. **The Model.** Let $I_t \geq 0$ be the inventory of a given commodity at time t, and let $\delta \geq 0$ be the instantaneous rate of stock deterioration. The cost of storing I_t units of goods from time t to time $t+1$, paid at time t, is kI_t, where $k \geq 0$. Let Y_t be the output of the commodity. Let X_t be the total available supply at time t, which takes values in $\mathsf{X} := [b, \infty)$, where $b \in \mathbb{R}$, and is defined by

$$X_t := \mathrm{e}^{-\delta} I_{t-1} + Y_t. \tag{1}$$

Let $p \colon \mathsf{X} \to \mathbb{R}$ be the inverse demand function. We assume that p is continuous, strictly decreasing, and bounded above.[6] Let P_t be the market price at time t. Without inventory, $P_t = p(Y_t)$. In general, market equilibrium requires that total supply equals total demand (sum of the consumption and the speculation demand), equivalently,

$$X_t = p^{-1}(P_t) + I_t. \tag{2}$$

An immediate implication of (2) is that $P_t \leq p(b)$ and

$$P_t \geq p(X_t), \quad \text{with equality holding when } I_t = 0. \tag{3}$$

Let M_{t+1} be the real one-period stochastic discount factor applied by investors at time t. The price process $\{P_t\}$ is restricted by

$$P_t \geq \mathrm{e}^{-\delta}\, \mathbb{E}_t\, M_{t+1} P_{t+1} - k, \quad \text{with equality holding if } I_t > 0 \text{ and } P_t < p(b). \tag{4}$$

In other words, per-unit expected discounted returns from storing the commodity over one period cannot exceed the per-unit cost of taking that position.

[6] We impose an upper bound to simplify exposition. In Appendix A we show that unbounded demand functions can also be treated, and the theory below still holds.

Combining (3) and (4) yields[7]

$$P_t = \min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_t\, M_{t+1}P_{t+1} - k,\ p(X_t)\right\},\ p(b)\right\}. \tag{5}$$

Both $\{M_t\}$ and $\{Y_t\}$ are exogenous, obeying

$$M_t = m(Z_t, \varepsilon_t) \quad \text{and} \quad Y_t = y(Z_t, \eta_t), \tag{6}$$

where m and y are Borel measurable functions satisfying $m \geq 0$ and $y \geq b$, $\{Z_t\}$ is a time-homogeneous irreducible Markov chain (possibly multi-dimensional) taking values in a finite set Z, and the innovations $\{\varepsilon_t\}$ and $\{\eta_t\}$ are IID and mutually independent.

Example 2.1. The setup in (6) is very general and allows us to model both correlated and uncorrelated $\{M_t, Y_t\}$ processes. In particular, it does not impose that $\{M_t\}$ and $\{Y_t\}$ are driven by a *common* Markov process, nor does it restrict that they are mutually dependent. Consider for example $Z_t = (Z_{1t}, Z_{2t})$, where $\{Z_{1t}\}$ and $\{Z_{2t}\}$ are mutually independent, possibly multi-dimensional Markov processes, and $M_t = m(Z_{1t}, \varepsilon_t)$ and $Y_t = y(Z_{2t}, \eta_t)$. In this case, $\{M_t\}$ and $\{Y_t\}$ are mutually independent, although they are autocorrelated. If in addition $\{Z_{1t}\}$ (resp., $\{Z_{2t}\}$) is IID or does not exist, then $\{M_t\}$ (resp., $\{Y_t\}$) is IID. Obviously, these are all special cases of (6). More examples are given in Section 3 below.

Below, the next-period value of a random variable X is denoted by $\hat{X}$. In addition, we define $\mathbb{E}_z := \mathbb{E}\,(\,\cdot \mid Z = z)$ and assume throughout that

$$\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}p(\hat{Y}) - k > 0 \ \text{ for all } \ z \in \mathsf{Z}. \tag{7}$$

In other words, the present market value of future output covers the cost of storage.

2.2. **Discounting.** To discuss conditions under which price equilibria exist, we need to jointly restrict discounting and depreciation. To this end, we introduce the quantity[8]

$$\kappa(M) := \lim_{n\to\infty} \frac{-\ln q_n}{n} \quad \text{where} \quad q_n := \mathbb{E}\prod_{t=1}^{n} M_t. \tag{8}$$

To interpret $\kappa(M)$, note that, in this economy, $q_n(z) := \mathbb{E}_z \prod_{t=1}^n M_t$ is the state z price of a strip bond with maturity n. Since $\{Z_t\}$ is irreducible, initial conditions do not determine long-run outcomes, so $q_n(z)$ is approximately constant at q_n defined in (8) when n is large.

---

[7]The minimization over $p(b)$ in (5) is required due to the generic stochastic discounting setup. As can be seen below, our theory allows for large and highly persistent discounting process (e.g., arbitrarily long sequences of negative low interest rates under risk neutrality), in which case $\mathrm{e}^{-\delta}\,\mathbb{E}_t\, M_{t+1} > 1$ with positive probability, thus the marginal reward of speculation, $\mathrm{e}^{-\delta}\,\mathbb{E}_t\, M_{t+1}P_{t+1} - k$, can be larger than $p(b)$. The extra minimization operation is then required to meet the equilibrium condition $P_t \leq p(b)$.

If $\mathrm{e}^{-\delta}\,\mathbb{E}_t\, M_{t+1}P_{t+1} - k > p(b)$, the equilibrium condition implies that $P_t \equiv p(b) < \mathrm{e}^{-\delta}\,\mathbb{E}_t\, M_{t+1}P_{t+1} - k$. In essence, price reaches its upper bound, and the non-arbitrage condition is violated. As a consequence, investors are incentivized to maintain inventories to exploit arbitrage opportunities. If this pattern persists, investments in inventory could grow arbitrarily large.

[8]Here and below, expectation without a subscript refers to the stationary process, where Z_0 follows the (necessarily unique) stationary distribution.

As a result, we can interpret $\kappa(M)$ as the asymptotic yield on risk-free zero-coupon bonds as maturity increases without limit.

In Lemma A.1 of Appendix A, we provide a numerical method for calculating $\kappa(M)$ by connecting it to the spectral radius of a discount operator.

Assumption 2.1. $\kappa(M) + \delta > 0$.

Assumption 2.1 is analogous to the classical condition $r + \delta > 0$ found in constant interest rate environment of Deaton and Laroque (1996) and many other studies.[9] In the more general setting we consider, Assumption 2.1 ensures sufficient discounting, adjusted by the depreciation rate, to generate finite prices in the forward-looking recursion (5), while still allowing for arbitrarily long sequences of negative yields in realized time series.

2.3. **Equilibrium.** We take (X_t, Z_t) as the state vector, taking values in $\mathsf{S} := \mathsf{X} \times \mathsf{Z}$. We assume free disposal as in Cafiero et al. (2015) to ensure that the equilibrium prices are non-negative. Conjecturing that a stationary rational expectations equilibrium exists and satisfies (5), an *equilibrium pricing rule* is defined as a function $f^* \colon \mathsf{S} \to \mathbb{R}_+$ satisfying

$$f^*(X_t, Z_t) = \min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_t\, M_{t+1} f^*(X_{t+1}, Z_{t+1}) - k,\; p(X_t)\right\},\; p(b)\right\}$$

with probability one for all t, where X_{t+1} is defined by (1) and, recognizing free disposal, storage therein is determined by $I_t = i^*(X_t, Z_t)$, where $i^* : \mathsf{S} \to \mathbb{R}_+$ is the *equilibrium storage rule*[10]

$$i^*(x,z) := \begin{cases} x - p^{-1}[f^*(x,z)], & \text{if } x < x^*(z) \\ x^*(z) - p^{-1}(0), & \text{if } x \geq x^*(z) \end{cases} \tag{9}$$

with

$$x^*(z) := \inf\left\{x \in \mathsf{X} : f^*(x,z) = 0\right\}.$$

Let $\mathscr{C}$ be the space of bounded, continuous, and non-negative functions f on S such that $f(x,z)$ is decreasing in x, and $f(x,z) \geq p(x)$ for all (x,z) in S. Given an equilibrium pricing rule f^*, let

$$\bar{p}(z) := \min\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\, \hat{M} f^*(\hat{Y}, \hat{Z}) - k,\; p(b)\right\}.$$

The next theorem provides conditions under which the equilibrium pricing rule exists, is uniquely defined, and gives a sharp characterization of its analytical properties.

Theorem 2.1 (Existence and Uniqueness of Equilibrium Price). *If Assumption 2.1 holds, then a unique equilibrium pricing rule f^* exists in $\mathscr{C}$. Furthermore,*

(i) $f^*(x,z) = p(x)$ *if and only if* $x \leq p^{-1}[\bar{p}(z)]$,

(ii) $f^*(x,z) > \max\{p(x), 0\}$ *if and only if* $p^{-1}[\bar{p}(z)] < x < x^*(z)$,

(iii) $f^*(x,z) = 0$ *if and only if* $x \geq x^*(z)$, *and*

[9] In the model with constant risk-free rate r, the discount rate M_t is $1/(1+r)$ at each t, so, by the definition in (8), we have $\kappa(M) = \lim_{n\to\infty} n\ln(1+r)/n = \ln(1+r) \approx r$.

[10] Throughout, we adopt the usual convention that $\inf \emptyset = \infty$.

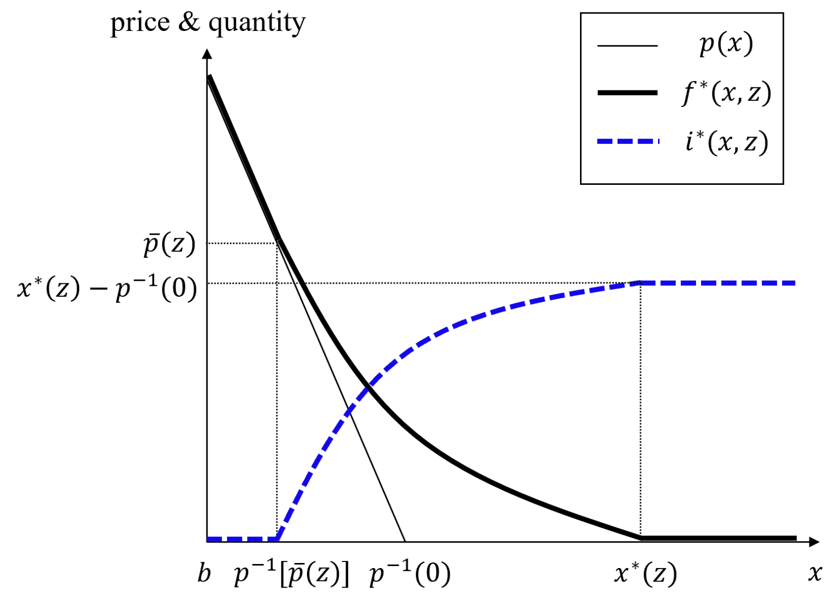

FIGURE 2. Illustration of the equilibrium price f^* and the equilibrium storage i^*. Here p is the inverse demand function, $\bar{p}$ is the decision threshold for speculators to start holding inventories, and x^* is the free-disposal threshold.

(iv) *$f^*(x,z)$ is strictly decreasing in x when it is strictly positive and* $e^{-\delta}\,\mathbb{E}_z\,\hat{M} < 1$.

In Appendix A, we show that the equilibrium pricing rule is the unique fixed point of an operator defined by the equilibrium conditions (named as the equilibrium price operator) and can be solved for via successive approximation. In particular, the equilibrium price operator is an *eventual contraction mapping* on a suitably constructed candidate space (which reduces to $\mathscr{C}$ when the demand function is bounded). This guarantees existence, uniqueness, and computability of the equilibrium solutions.[11] In Online Appendix **??**, we provide an endogenous grid algorithm that solves for the equilibrium objects efficiently. In Online Appendix **??**, we show that, in some rather standard settings, Assumption 2.1 is necessary as well as sufficient: no equilibrium price sequence exists if Assumption 2.1 fails.[12]

The next result states the properties of the equilibrium storage rule.

[11]Since we allow for arbitrarily long sequences of negative yields, it is challenging to construct operators that contract in one step, since the one-period yield, which is typically required to construct the modulus of contraction, can be overly small, violating Blackwell (1965)'s sufficient conditions for contraction. To solve this problem, Assumption 2.1 bounds the asymptotic yield instead of the one-period yield, allowing us to construct an n-step contraction.

[12]See Proposition **??** in Online Appendix **??**. Note that Theorem 2.1 establishes existence and uniqueness of stationary equilibria. However, this does not preclude the possibility of nonstationary equilibria, including nonstationary price sequences and bubbles. For some recent literature on bubbles in asset markets, see Barlevy (2012), Guerron-Quintana et al. (2023), Plantin (2023), and Hirano and Toda (2024).

Proposition 2.1 (Existence and Uniqueness of Equilibrium Storage). *If Assumption 2.1 holds, then the equilibrium storage rule $i^*(x,z)$ is increasing in x and continuous. Furthermore,*

(i) $i^*(x,z) = 0$ *if and only if* $x \leq p^{-1}[\bar{p}(z)]$,

(ii) $0 < i^*(x,z) < x^*(z) - p^{-1}(0)$ *if and only if* $p^{-1}[\bar{p}(z)] < x < x^*(z)$,

(iii) $i^*(x,z) = x^*(z) - p^{-1}(0)$ *if and only if* $x \geq x^*(z)$, *and*

(iv) $i^*(x,z)$ *is strictly increasing in* x *if* $p^{-1}[\bar{p}(z)] < x < x^*(z)$ *and* $\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} < 1$.

Proposition 2.1 indicates that speculators hold inventories if and only if the market value of the total available supply $p(x)$ is below the decision threshold $\bar{p}(z)$. Otherwise, selling all commodities at hand is optimal, in which case the equilibrium price is $f^*(x,z) = p(x)$. The equilibrium price and storage properties are illustrated in Figure 2 under a linear demand function. The equilibrium rules are sketched for a given exogenous state z.

3. Interest Rates and Prices: Theoretical Results

Next we inspect the relationship between interest rates and commodity prices implied by the model. To this end, we assume that speculators discount future payoffs according to market prices:

$$M_t = \frac{1}{R_t}, \quad \text{where} \quad R_t := r(Z_t, \varepsilon_t).$$

Hence r is a real-valued non-negative Borel measurable function of the state process and innovation ε. The process $\{R_t\}$ is interpreted as the gross real interest rate on risk-free bonds. We therefore preserve the risk-neutrality assumption of the standard competitive storage model, while allowing the risk-free rate to be state-dependent. Throughout this section, we impose the assumptions of Section 2.

3.1. **Correlations.** We first explore general conditions under which interest rates and commodity prices are negatively correlated. As a first step, we state a finding concerning monotonicity of equilibrium objects with respect to the exogenous states.

Proposition 3.1 (Monotonicity of Equilibrium Objects w.r.t. the Exogenous State). *If $r(z,\varepsilon)$ and $y(z,\eta)$ are nondecreasing in z, and $\{Z_t\}$ is a monotone Markov process,*[13] *then the equilibrium pricing rule $f^*(x,z)$, the equilibrium inventory $i^*(x,z)$, and the decision threshold $\bar{p}(z)$ are all decreasing in z.*

The intuition is as follows: If (i) a higher Z_t shifts up the distribution of Z_{t+1} in terms of first-order stochastic dominance and (ii) interest rates and output are both nondecreasing in this state variable, then a high Z_t today tends to generate both sustained high output and more impatient speculators in the future. The former boosts supply, while the latter diminishes the incentive for holding inventories, reducing speculative demand. As a result, both inventories and prices are lower.

[13] Here monotonicity is defined in terms of first-order stochastic dominance. See Appendix B for its formal definition.

The assumptions of Proposition 3.1 do not restrict R_t and Y_t to be strictly increasing in Z_t, nor do they impose that R_t and Y_t are driven by a common factor. In particular, the second assumption concerning monotone Markov process is standard (see Appendix B for sufficient conditions). Below, we discuss the first assumption through examples.

Example 3.1. If $\{R_t\}$ and $\{Y_t\}$ are IID and mutually independent, then we can set $Z_t \equiv 0$, $\varepsilon_t = R_t$ and $\eta_t = Y_t$, in which case $r(z, \varepsilon) = \varepsilon$ and $y(z, \eta) = \eta$. Hence, the first two assumptions of Proposition 3.1 hold.

Example 3.2. If $\{R_t\}$ and $\{Y_t\}$ are autocorrelated and mutually independent, then we can write Z_t as $Z_t = (Z_{1t}, Z_{2t})$, where $\{Z_{1t}\}$ and $\{Z_{2t}\}$ are mutually independent, possibly multi-dimensional Markov chains, and $R_t = r(Z_{1t}, \varepsilon_t)$ and $Y_t = y(Z_{2t}, \eta_t)$. In this case, the first assumption of Proposition 3.1 holds as long as r is nondecreasing in Z_{1t} and y is nondecreasing in Z_{2t}.

Example 3.3. If $\{R_t\}$ and $\{Y_t\}$ are finite Markov processes, then we can set $\varepsilon_t = \eta_t \equiv 0$ and define $Z_t = (R_t, Y_t)$, in which case the first assumption of Proposition 3.1 holds automatically, while the second assumption holds as long as $\{R_t\}$ and $\{Y_t\}$ are monotone and non-negatively correlated Markov processes.

We can now state our main result concerning correlation. In doing so, we suppose that $Z_t = (Z_{1t}, \dots, Z_{nt})$ takes values in $\mathbb{R}^n$.

Proposition 3.2 (**Negative Correlation of Interest Rates and Prices**). *If the conditions of Proposition 3.1 hold and $\{Z_{1t}, \dots, Z_{nt}\}$ are independent for each fixed t, then*

$$\mathrm{Cov}_{t-1}(P_t, R_t) \leq 0 \quad \textit{for all } \; t \in \mathbb{N}.$$

As Proposition 3.1 illustrates, when interest rates and output are both positively affected by the monotone exogenous state process, commodity prices will be negatively affected by the exogenous state. Therefore, there is a trend of comovement (in opposite directions) between commodity price and interest rate, resulting in a negative correlation. The proof of Proposition 3.2 relies on the Fortuin–Kasteleyn–Ginibre inequality.

Note that the independence-across-dimensions condition in Proposition 3.2 cannot be omitted. In Appendix C, we provide examples showing that if $\{Z_{1t}, \dots, Z_{nt}\}$ are positively or negatively correlated for some $t \in \mathbb{N}$, then interest rates and prices can be positively correlated. This is because contemporaneous correlation across dimensions of Z_t can alter comovement of interest rates and commodity prices. (Such correlation can either strengthen or weaken the impact of interest rates on commodity prices, yielding rich model dynamics.)

Remark 3.1. In Appendix B, we show that Proposition 3.2 can be extended to the setting of Section 2, where agents are not necessarily risk neutral. In particular, $\mathrm{Cov}_{t-1}(P_t, M_t) \geq 0$ holds.

Example 3.4. (The Speculative Channel). In applications $\{R_t\}$ typically follows a Markov process, while $\{Y_t\}$ represents a sequence of supply shocks (e.g., harvest failures, conflicts around oil production sites, so on), which is IID and less likely to be affected by the monetary conditions (see, e.g., Deaton and Laroque, 1992; Cafiero et al., 2015). Hence, $\{R_t\}$ and $\{Y_t\}$ are mutually independent. In this case, all the effects of interest rates on commodity prices transit through commodity speculation. By letting $Z_t = R_t$, $\varepsilon_t \equiv 0$ and $\eta_t = Y_t$, we have $r(z, \varepsilon) = z$ and $y(z, \eta) = \eta$. Hence, all the assumptions of Proposition 3.2 hold as long as $\{R_t\}$ is a finite monotone Markov process (e.g., a discrete version of a positively correlated AR(1) process) and Assumption 2.1 holds (see the next section). In this case, Proposition 3.2 implies that interest rates are negatively correlated with commodity prices, which matches the empirical results of Frankel (1986, 2008a, 2014).

Example 3.5. (The Global Demand Channel). Since the output of the commodity, Y_t, enters linearly in total availability, it can be redefined as a linear combination of two shocks: $Y_t = Y_t^S - Y_t^D$, where Y_t^S is the supply shock and Y_t^D is the demand shock. Hence Y_t can be interpreted as a net supply shock. There is widespread evidence that both types of shocks matter in commodity markets, albeit with relative importance depending on the commodities (see, e.g., Kilian, 2009; Gouel and Legrand, 2022). Unlike supply shocks, demand shocks are likely to be affected by monetary policies. Since interest rates affect global demand (Ramey, 2016), an interest rate shock leads to an aggregate demand shock that affects all commodities.[14] If interest rates follow a Markov process, it implies that $Z_t = (R_t, Z_{2t})$ and $Y_t = y(Z_{2t}, \eta_t)$, where $\{Z_{2t}\}$ is a Markov process that is correlated with $\{R_t\}$. Hence, Z_t is contemporaneously correlated and the independence-across-dimensions condition of Proposition 3.2 fails. However, the theory of Section 2 still applies and can be used to quantify the impact of interest rates on commodity prices.

3.2. **Causality.** We now study the causal relationship between interest rates and commodity prices. As a first step, we state an elementary monotonicity property concerning interest rates and prices. To this end, we take $\{R_t^i\}$ to be the interest rate process for economy $i \in \{1, 2\}$. In addition, let f_i^* and $\{P_t^i\}$ be the equilibrium pricing rule and the price process corresponding to $\{R_t^i\}$.

Proposition 3.3 (Causal Effect of Ordered Interest Rates on Prices). *If $R_t^2 \leq R_t^1$ with probability one for all $t \geq 0$, then $f_1^* \leq f_2^*$ and $P_t^1 \leq P_t^2$ with probability one for all t.*

The intuition is straightforward. Seen from the speculative channel, lower interest rates reduce the opportunity cost of storage. Lower storage costs encourage a build-up of inventories. Higher demand for inventories induces higher prices.

Proposition 3.3 has limited implications because it concerns variations in interest rates that are uniformly ordered over time. Next, we aim to relax this assumption.

[14]Even if the demand shock is an aggregate shock affecting all commodities, its impact is likely to vary across different commodities. For instance, demand for food commodities may exhibit lower sensitivity to GDP shocks due to their low income elasticity, whereas metals and energy commodities may respond more significantly due to their utilization as intermediate inputs.

Let $\{X_t^i\}$ and $\{Z_t^i\}$ be respectively the endogenous and exogenous state processes for economy $i \in \{1, 2\}$. Unless otherwise specified, we assume that both economies experience the same innovation process $\{\eta_t\}$ to output.

Proposition 3.4 (Causal Effect of Interest Rates on Prices). *Suppose $\{R_t\}$ is a monotone finite-state Markov process and $Y_t = y(R_t, \eta_t)$, where y is nondecreasing in R. If $X_{t-1}^2 \leq X_{t-1}^1$, $R_{t-1}^1 \leq R_{t-1}^2$ and $R_t^2 \leq R_t^1$ with probability one, then $P_t^1 \leq P_t^2$ with probability one.*

Proposition 3.4 indicates that if the interest rate is a monotone finite Markov process that has a nonnegative effect on output, then, conditional on the same previous state, a higher interest rate today reduces commodity price in the same period.

Proposition 3.5 (Causal Effect of Interest Rates on Prices Over Time). *Suppose $\{R_t\}$ is a monotone finite-state Markov process and $Y_t = y(R_t)$, where y is nondecreasing. If $X_{t-1} \leq X_t$, $R_{t-1} \geq R_t$ and $R_t \leq R_{t+1}$ with probability one, then $P_t \geq P_{t+1}$ with probability one.*

Proposition 3.5 above indicates that, if R_t is a monotone finite Markov process and has a nonnegative effect on Y_t, X_t is no less than its previous period level, and R_t is no higher than its previous period level, then an increase in interest rates next period causes falling commodity prices.

4. Quantitative Analysis

To illustrate the quantitative implications of our theory, we study the impact of interest rates on commodity prices through two channels: the speculative and the aggregate demand channels. To this end, we use a stylized model that requires a minimum number of parameters to characterize its behavior.[15] We calibrate the model to a quarterly setting to limit the number of state variables.[16]

The main takeaways from this section are fourfold. First, impulse response functions (IRFs) show that prices decrease immediately following a positive interest rate shock and slowly converge to their long-run average, with a stronger decrease and a slower convergence when the global demand channel is active. Second, inventory dynamics are ambiguous: while inventories tend to decline after an interest rate shock due to reduced stockpiling incentives from higher interest rates, they increase after an initial decline when the demand channel is active. This is because reduced demand lowers prices, creating profit opportunities for storage. Overall, inventories tend to return to their long-run average at an even slower pace than prices. Third, price volatility exhibits sensitivity to inventory dynamics: a larger response in inventory tends to generate an inversely larger response in price volatility. Fourth, the strength of these IRFs is highly state-dependent, being more pronounced for high availabilities.

[15] Gouel and Legrand (2022) show that fitting most moments of a commodity market necessitates a rich storage model with supply reaction, autocorrelated shocks, and news shocks. Since most of these elements are specific to each commodity market and are orthogonal to the question studied here, we abstract for them and focus on a model with minimal free parameters.

[16] A monthly real interest rate process requires a rich autoregressive structure, introducing many lags.

4.1. **Specification.** For simulating the model, we adopt a linear demand function

$$p(x) = \bar{p}\left[1 + (x/\mu_Y - 1)/\lambda\right], \tag{10}$$

where $\bar{p} > 0$ is the steady-state price,[17] $\mu_Y > 0$ is the mean of the commodity output process (so also the steady-state consumption level), and $\lambda < 0$ is the price elasticity of demand.[18] We assume that all storage costs are related to depreciation (i.e., $k = 0$ and $\delta \geq 0$). As Gouel and Legrand (2022) show, when calibrated to represent the same proportion of the steady-state price, these two types of storage costs have indistinguishable effects on price moments, so focusing only on one involves no loss of generality.

We assume that the annual interest rate, measured at a quarterly frequency, follows the first-order autoregressive process

$$R_t^a = \mu_R(1-\rho_R) + \rho_R R_{t-1}^a + \sigma_R\sqrt{1-\rho_R^2}\,\varepsilon_t^R, \quad \{\varepsilon_t^R\} \overset{\text{IID}}{\sim} N(0,1). \tag{11}$$

We follow Example 3.5 and consider that $\{Y_t\}$ represents a net supply shock: $Y_t = Y_t^S - Y_t^D$. Commodity output, $\{Y_t^S\}$, follows a truncated normal distribution with mean μ_Y, standard deviation $\mu_Y\sigma_Y$, truncated at 5 standard deviations. The truncation of the distribution (also adopted, inter alia, in Deaton and Laroque, 1992) ensures a lower bound for commodity output and total available supply. Commodity demand is proportional to economic activity: $Y_t^D = \alpha A_t$. Economic activity is represented by the following simple IS curve:

$$A_t = \rho_A A_{t-1} - \gamma(R_t^a - \mu_R), \tag{12}$$

where $|\rho_A| < 1$ parameterizes the persistence of economic activity and $\gamma \geq 0$ the effect of interest rate deviation from the mean on it. For constant interest rates at the mean, economic activity is just 0.[19]

To simplify the problem, we have assumed that A_t is driven only by interest rates and by its own persistence. These assumptions avoid the need to identify the innovation process of economic activity and, more importantly, to represent the effect of economic activity on interest rates. The joint dynamic of (R_t^a, A_t) is represented by a SVAR(1) model:

$$\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}\begin{bmatrix} R_t^a \\ A_t \end{bmatrix} = \begin{bmatrix} \mu_R(1-\rho_R) \\ \gamma\mu_R \end{bmatrix} + \begin{bmatrix} \rho_R & 0 \\ 0 & \rho_A \end{bmatrix}\begin{bmatrix} R_{t-1}^a \\ A_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_R\sqrt{1-\rho_R^2} \\ 0 \end{bmatrix}\varepsilon_t^R. \tag{13}$$

To make this process compatible with our assumptions, we discretize it.

This setup is a special case of the theoretical framework established in Section 3, with $Z_t = (R_t^a, A_t)$, $\varepsilon_t \equiv 0$, $\eta_t = Y_t^S$, and $y(Z_t, \eta_t) = Y_t^S - \alpha A_t$. We use an annual interest rate process to obtain results that are directly comparable to others in the literature, but the model calls for an interest rate at the quarterly frequency, so we define $R_t = r(Z_t, \varepsilon_t) = (R_t^a)^{1/4}$. Below we estimate the interest rate process and show that $\rho_R > 0$ (implying that

[17] If not otherwise specified, we designate by steady state the equilibrium in the absence of any shocks.

[18] An isoelastic inverse demand function has also been tested and the results are robust to this change.

[19] We assume a contemporaneous effect of interest rate on economic activity to be consistent with recent VAR results that point to such effects (Bauer and Swanson, 2023).

$\{R_t^a\}$ is a monotone Markov process) and that the discount condition in Assumption 2.1 holds. Hence, all the statements of Theorem 2.1 and Proposition 2.1 are valid.

In a first step, we will analyze the speculative channel without the global demand channel (so assuming $\alpha = 0$). In this setting, $r(z, \varepsilon)$ and $y(z, \eta)$ are increasing in z. So, Proposition 3.1 is valid, and since in this case $\{R_t\}$ is independent of $\{Y_t\}$, the assumptions (and thus conclusions) of Proposition 3.2 also hold. Propositions 3.1 and 3.2 do not hold in general with the global demand channel.

This choice of parameterization limits the free parameters that matter in the analysis of price movements to δ, λ, α, and σ_Y. Indeed, the interest rate process is estimated on observations, the economic activity process is calibrated based on Bauer and Swanson (2023), and we can normalize $\bar{p}$ and μ_Y to unity, since their effect is only to set the average price and quantity levels. To ease interpretation and limit the number of parameters to adjust, we fix σ_Y to 0.05.[20] If only the speculative channel is active, this choice is innocuous as we can prove that adjusting the intercept and slope of the demand function is equivalent to adjusting the mean and variance of the output process (see the proof in Online Appendix **??**, which is a generalization of Proposition 1 of Deaton and Laroque, 1996).

To calibrate the real interest rate process $\{R_t^a\}$, we follow the literature on interest rates and commodity prices (e.g., Frankel, 2008a; Gruber and Vigfusson, 2018; Kilian and Zhou, 2022) and use the nominal one-year treasury yield. We deflate this rate by a measure of expected inflation. Note that this choice of interest rate slightly reduces the influence of the zero lower bound compared to the 3-month treasury yield, making it a better measure of monetary policy. Expected inflation is calculated through an autoregressive model estimated on a 30-year window prior to the year of interest to account for changes in the dynamics of inflation.[21] This is the real interest rate represented in Figure 1. The maximum likelihood estimation of (11) over the period 1962–2022 yields $\mu_R = 1.0062$, $\rho_R = 0.9407$, and $\sigma_R = 0.03$.

These results imply that the stationary mean of the real interest rate process is about 0.6%, with an unconditional standard deviation of about 3%. Note that any constant spread above the risk-free rates can be captured in our model by δ. Therefore, when interpreting the values of δ in what follows, it should be kept in mind that δ represents storage costs, a premium above risk-free rates, and any long-run trend in commodity prices.[22]

To calibrate the economic activity process, we use proxy SVAR estimates from Bauer and Swanson (2023). Our economic activity process presents two free parameters: ρ_A and γ. We calibrate them by matching two moments: the number of months needed to attain the minimum level of activity after a monetary shock and the size of the decrease at

[20] According to Gouel and Legrand (2022), a coefficient of variation of 5% for the net supply shock is slightly above the total shock (demand plus supply) affecting the aggregate crop market of maize, rice, soybeans, and wheat, but below the shocks affecting each of these markets individually.

[21] Using lagged inflation or ex-post inflation would lead to a similar real interest rate process.

[22] See Bobenrieth et al. (2021) for an analysis of the role of commodity price trends in the storage model.

the minimum. This calibration is done conditional on the interest rate process calibrated previously. The preferred estimation of Bauer and Swanson (2023, Figure 8) indicates 9 months to reach a decrease of industrial production of -0.4% after a 25 bp monetary shock. In our setting, this leads to $\rho_A = 0.52$ and $\gamma = 0.95$.

If only the speculative channel is active, we discretize the interest rate process (11) into an N-state Markov chain using the method of Tauchen (1986) with $N = 101$. If both channels are active, we transform the SVAR(1) model of equation (13) into a VAR(1) model and discretize it using the approach of Schmitt-Grohé and Uribe (2014).

In our model, demand is specified as $p^{-1}(P_t) + Y_t^D$. With the steady-state value of $p^{-1}(P_t)$ normalized to unity and Y_t^D having a zero mean, the parameter Y_t^D may be viewed as a deviation from the steady-state demand level. Within this framework, the parameter α plays a crucial role in dictating the extent to which shocks in economic activity influence the commodity market. For our central calibration, we choose $\alpha = 0.2$ to align with the immediate price reactions of a commodity index to positive interest rate shocks, as identified in the IRFs reported in Bauer and Swanson (2023, Figure 8). However, to acknowledge the variability in the responsiveness of different commodities to economic conditions and ensure robustness of our findings, we conduct a series of simulations wherein we systematically vary the value of α, thoroughly investigating its impact across various scenarios.

Our first step is to verify Assumption 2.1, which requires $\kappa(M) > -\delta$. Figure 3 plots $\kappa(M)$ calculated at different $(\mu_R, \rho_R, \sigma_R)$ values.[23] In the left panel, we fix ρ_R at its estimated value and create a contour plot of $\kappa(M)$ for (μ_R, σ_R). In the right panel, we fix σ_R at its estimated value and plot $\kappa(M)$ as a function of (ρ_R, μ_R). The figure shows that $\kappa(M)$ is increasing in μ_R and decreasing in σ_R and ρ_R. In general, $\kappa(M) > -\delta$ fails only when μ_R is sufficiently low, or when ρ_R or σ_R is very large. The black solid curves represent the thresholds at which $\kappa(M) = -0.02, -0.01, 0$, respectively. Clearly, Assumption 2.1 holds at the estimated $(\mu_R, \rho_R, \sigma_R)$ values even when $\delta = 0$.

Having verified Assumption 2.1, we solve for the equilibrium pricing rule using the following methods. Expectation terms are replaced by simple sums using the exogenous state Markov chain and a 7-point Gaussian quadrature for the output process. Starting from a guessed initial solution, the pricing rule is found by iterating with the equilibrium price operator, which is globally convergent by Theorem A.1 in the appendix. To maximize efficiency, we apply a modified version of the endogenous grid method (Carroll, 2006).[24] Details of the algorithm and computation are given in Online Appendix **??**.[25]

4.2. **Experiments.** We begin the quantitative analysis by examining the impact of the speculative channel on the commodity price dynamics (so assuming $\alpha = 0$). We then investigate the role of the global demand channel. While speculative incentives remain

[23]The method for computing $\kappa(M)$ is described in Lemma A.1 of Appendix A.

[24]We use a 100-point exponential grid for I_t in the range of $[0, 2]$ with median value 0.5. Function approximation is implemented via linear interpolation. We terminate the iteration process at precision 10^{-4}.

[25]Replication materials are available at DOI: 10.57745/JV1JR6.

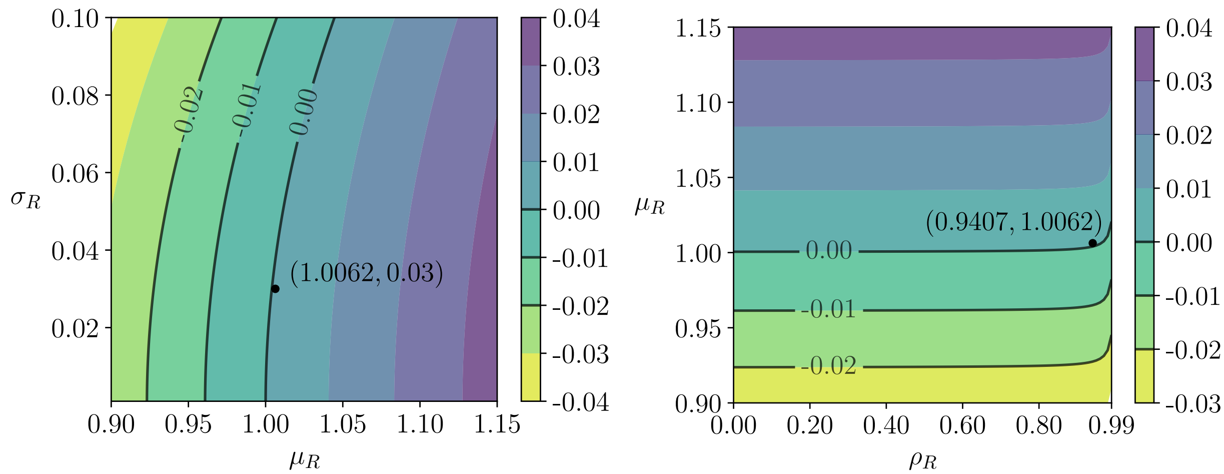

FIGURE 3. $\kappa(M)$ values under different $(\mu_R, \rho_R, \sigma_R)$. Here $\kappa(M)$ represents the asymptotic yield on risk-free zero-coupon bonds as maturity increases without limit, and $(\mu_R, \rho_R, \sigma_R)$ governs the gross real interest rate process.

active under the global demand channel (for example, via autocorrelation of economic activity, which affects the speculators' expectations), we can isolate and analyze the effects through each channel by comparing the IRFs.

4.2.1. *Speculative channel.* Unless otherwise specified, we assume $\delta = 0.02$ and $\lambda = -0.06$. This combination of parameters leads to the following price moments on the asymptotic distribution: a coefficient of variation of 24%, a first-order autocorrelation of 0.61, and a skewness of 2.9, all of which align with the empirical observations.[26]

Since the Markov process we adopt here is symmetric around the mean, the pricing rule fluctuates symmetrically around the corresponding constant-discounting pricing rule. This has implications for unconditional price moments. Notably, the standard price moments of interest—the coefficient of variation, autocorrelation, and skewness—are nearly indistinguishable between this model and a constant-discounting model with the same average discounting (differing only at 3 digits). While this outcome may seem surprising, considering that the real rate volatility could be perceived as an additional source of volatility, in practice this does not create additional standard demand or supply shocks. When the interest rate falls below its mean, it prompts additional demand for storage, driving prices up. However, when the interest rate rises, these additional stocks are sold, exerting downward pressure on prices. These effects tend to offset each other. This indicates that the speculative channel may not contribute significantly to empirical analyses based on unconditional price moments.

However, the presence of time-varying interest rates holds significant implications for conditional moments, which could be exploited empirically. Below, we delve into this exploration using impulse response functions (IRFs). To capture the nonlinear dynamics

[26]See, e.g., Table V in Gouel and Legrand (2017) for measures of these moments for a sample of commodities.

generated by the storage model, we follow Koop et al. (1996) and define IRFs as state-and-history-dependent random variables. We calculate the IRFs to a 100 bp interest rate impulse (i.e., a 1% increase in the real interest rate). All IRFs represent percentage deviation from the benchmark simulation. A detailed discussion of the algorithm and computation is left to Online Appendix **??**.

Figure 4 shows the IRFs calculated at the stationary mean of (X_{t-1}, R^a_{t-1}). We first discuss the central IRFs corresponding to $\delta = 0.02$ and $\lambda = -0.06$, before analyzing the sensitivity of these IRFs to the parameters and states. The left panels present the IRFs for prices, which show an immediate price decrease followed by a gradual convergence to the long-run average over 2 to 4 years.[27] The middle panels display the IRFs for inventories, which, unlike prices, reach their lowest value more than a year after the shock, and even after 4 years, they have not returned to their long-run values. Finally, the right panels illustrate the IRFs for price volatility, namely, the conditionally expected standard deviation of price.[28] They indicate that price volatility largely follows stock dynamics with a peak reached after a year and an incomplete convergence after 4 years. This finding is consistent with the empirical results of Gruber and Vigfusson (2018), who show that higher interest rates imply higher price volatility.

The mechanism explaining this behavior is as follows. When the interest rate increases, speculators tend to dispose of stocks, which have become costlier to hold. This decreases current prices due to increased supply. However, the price decrease mitigates the extent of stock selling compared to what would happen if prices remained constant. In subsequent periods, stocks remain excessive due to persistent high opportunity costs, prompting speculators to continue selling them. Again, the price decrease acts as a cushion, preventing a complete sell-off of stocks. After more than a year of this dynamic, with increasingly smaller quantities being sold from inventories, interest rates have declined, alleviating the pressure to sell inventories, and prices are below their long-run values with anticipation of eventual convergence. Consequently, stock accumulation gradually increases as interest rates decline. In our illustration, since stock accumulation is slow, prices converge to their long-run values from below without overshooting, although this remains a possibility. After four years, prices have converged to their long-run values, as the only factor influencing the increase in stock levels is the reversion of the interest rate to its long-run average. As for price volatility, its behavior, opposite to that of inventories, can be explained by the fact that inventories serve as a key determinant of price volatility by providing a buffer against production shocks.

[27]These dynamics are very different from what would be expected from a transitory MIT shock to the interest rate in a standard storage model with a constant interest rate. In the latter scenario, an unexpected increase in the interest rate would also depress prices due to decreased stockholding. However, this price decrease would be short-lived, lasting only one period. With a transitory shock, the interest rate would revert to its baseline after one period, incentivizing stockpiling and consequently driving prices above their non-shock levels in all subsequent periods. This point is demonstrated in Online Appendix **??**, Proposition **??**.

[28]Price volatility is the square root of the conditional variance: $\sqrt{\mathbb{E}_{t-1}[f^*(X_t, Z_t)]^2 - [\mathbb{E}_{t-1} f^*(X_t, Z_t)]^2}$.

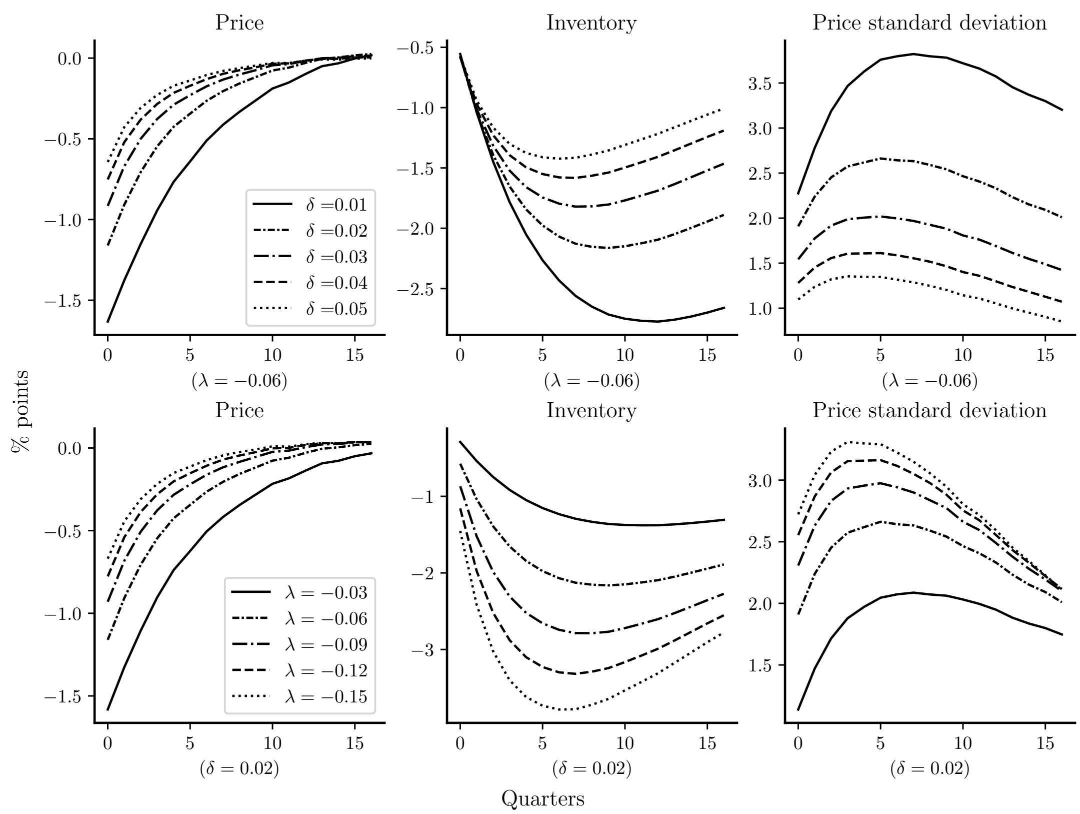

FIGURE 4. IRFs for a 100 bp real interest rate shock under different parameter setups without demand channel, fixing X_{t-1} and R^a_{t-1} at the stationary mean. Here X_{t-1} and R^a_{t-1} are respectively the total available supply and the annual gross real interest rate in the previous quarter, δ is the rate of depreciation, and λ is the demand elasticity.

Figure 4 also includes IRFs for various sets of parameters. The impact on commodity price increases as storage costs decrease: lower storage costs result in a higher initial decline and a slower return to equilibrium.[29] Intuitively, when storage costs are lower, opportunity costs become more significant and, as a result, variations in the interest rate have a greater influence on storage behavior and prices. This effect is evident in the observation that stocks decrease more sharply with an interest rate increase when storage costs are lower. In such scenarios, stock levels tend to be higher on average and more susceptible to changes in opportunity costs.

The price effects are more pronounced with a more inelastic demand function, as prices react significantly to variations in the sale of stock when demand is less elastic. Likewise, the decrease in stocks is less substantial when demand is more inelastic. This is due to the

[29]Price dynamics may not consistently show a less significant decrease with a higher storage cost when the global demand channel is activated, in which case storage costs have only a marginal and short-term effect on price dynamics.

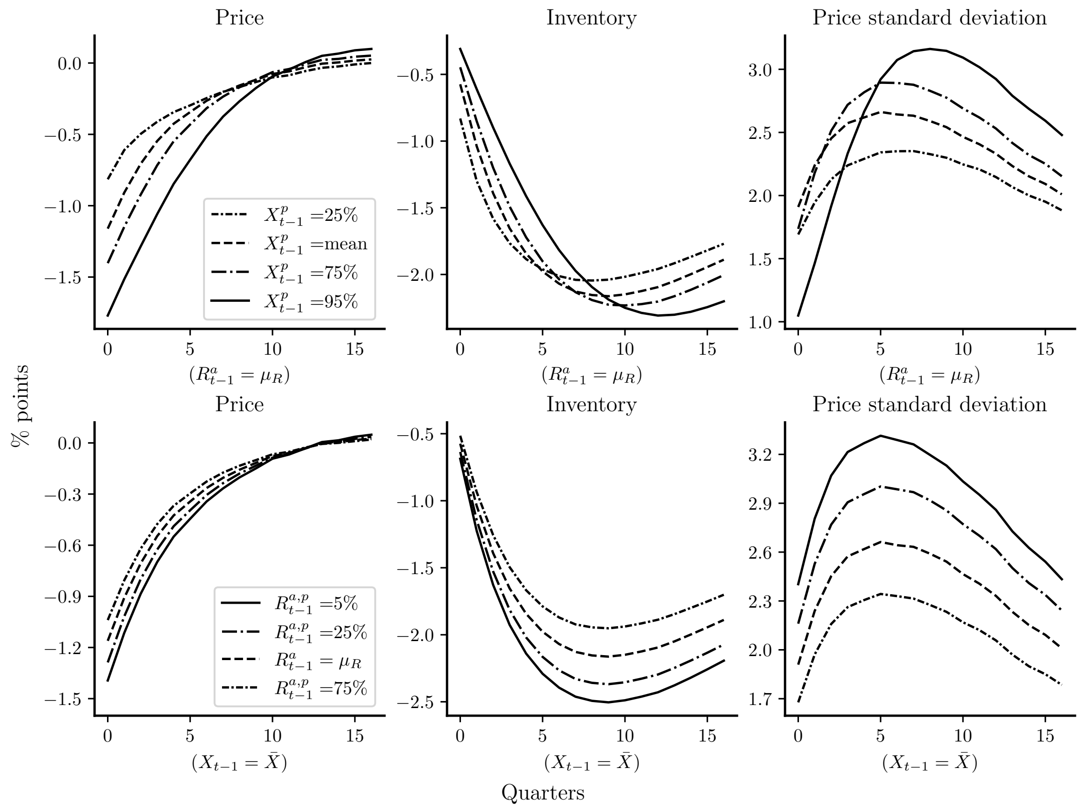

FIGURE 5. IRFs for a 100 bp real interest rate shock conditional on different states without demand channel. Here $(X^p_{t-1}, R^{a,p}_{t-1})$ denotes the percentile points of the realized total available supply and interest rate states on the stationary distribution, and μ_R and $\bar{X}$ are the stationary means of the interest rate and the total available supply processes, respectively.

fact that with inelastic demand, even a slight increase in sales from inventory can greatly depress prices, thereby reducing the incentive to excessively sell off stocks.

To explore the sensitivity of IRFs to states, Figure 5 draws the IRFs calculated for different realized values of (X_{t-1}, R^a_{t-1}). We use $(X^p_{t-1}, R^{a,p}_{t-1})$ to denote the percentile points of the realized (X_{t-1}, R^a_{t-1}) states on the stationary distribution. The top left panel shows that price responses are stronger when availability becomes larger. The immediate responses of price are respectively 1.72 and 2.17 times larger when availability increases from the 25% percentile to the 75% and 95% percentiles. This is because when availability is lower, inventory tends to be lower (Proposition 2.1), hence there is less room for stock adjustment and prices react much less in response to the interest rate shock. This intuition is verified by the top middle panel, which shows that a higher availability causes stock decumulation to last longer, yielding a larger decline in inventory in the medium to long run (in spite of a slightly lower immediate decline).

The bottom left panel in Figure 5 shows that price responses to a 100 bp interest rate shock tend to be slightly larger when interest rates are relatively lower. The overall trend of price and storage IRFs in the bottom panels of Figure 5 is consistent with our theory (Proposition 3.1), which shows that under lower interest rates, prices and inventories are in general higher and therefore more sensitive to variations in opportunity costs.[30]

Same as the previous cases, the right panels of Figure 5 show that the dynamics of price volatility are highly consistent with the inventory dynamics, with a larger response in speculative storage causing an oppositely larger response in price volatility.

4.2.2. *Global demand channel.* Next, we study the global demand channel as described in Section 4.1. A notable difference with the speculative channel is that the unconditional price moments are affected by the presence of the demand channel. This is due to the fact that the demand channel involves a demand shock driven by interest rates shocks, while the speculative channel implies offsetting demand and supply shocks. In addition, since the demand shock follows an autocorrelated process, it adds persistence to the price process. In the benchmark setting ($\alpha = 0.2$), for example, price autocorrelation increases from 0.61 to 0.88. Thus, unlike the speculative channel, the demand channel can play a role in econometric analysis of the storage model using unconditional price moments.[31] Nonetheless, the speculative channel, as well as the demand channel, could be empirically relevant for estimation strategies relying on conditional moments or unconditional moments calculated on quantities.

Figure 6 shows the IRFs with the global demand channel for various parameters. The impact of an interest rate shock on price is consistently more pronounced when the demand channel is activated. This is because the demand channel amplifies the price decline already induced by the speculative channel. In our main calibration, the dominance of the demand channel is evident, as the price effect is three times larger than under the speculative channel alone. Furthermore, since interest rates take a long time to converge back to their steady state, so does economic activity. As a consequence, the price effect also becomes more persistent.

The inventory dynamics differ notably with the inclusion of the demand channel. Initially, inventories experience a decline comparable in magnitude to that observed under the speculative channel alone. Subsequently, they rebound and tend to surpass their long-run level. This indicates that, after several quarters, the influence of low demand outweighs that of high interest rates. The low demand leads to suppressed prices with an expectation of future price recovery once demand is back to normal, incentivizing higher inventory holdings despite the disincentives posed by elevated interest rates.

[30]When the global demand channel is activated, market dynamics become more complex, in which case this monotone pattern can be disrupted due to global economic activity. We leave further investigation of this issue for future research.

[31]This potential role of the demand channel aligns with the findings of Gouel and Legrand (2022), which elucidates the empirical relevance of demand shocks and their role in explaining price autocorrelation.

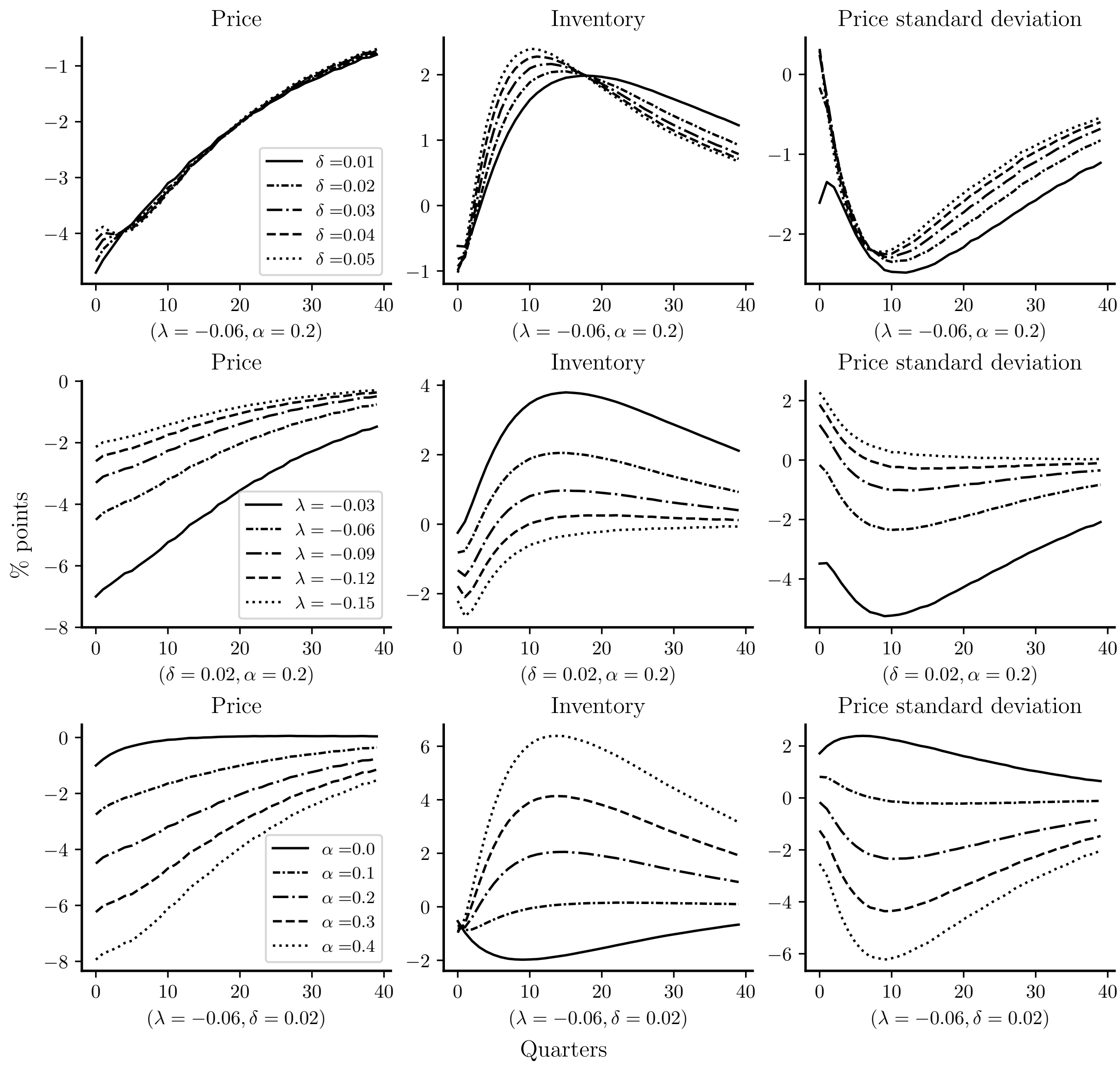

FIGURE 6. IRFs for a 100 bp real interest rate shock under different parameter setups, fixing X_{t-1} and R^a_{t-1} at the stationary mean. Here X_{t-1} and R^a_{t-1} are respectively the total available supply and the annual gross real interest rate in the previous quarter, δ is the rate of depreciation, λ is the demand elasticity, and α governs the sensitivity of the demand shock to economic activity.

These inventory dynamics also manifest in the behavior of price standard deviation, which moves inversely to inventory dynamics. Notably, it is important to recognize that the figure depicts the price standard deviation, rather than the coefficient of variation. Consequently, the price volatility is affected by the dynamics of expected prices as well.

Given that price volatility decreases less than the price level, the coefficient of variation actually experiences an increase over time.

The sensitivity of the IRFs to the model's parameters facilitates a deeper understanding of the mechanisms behind the demand channel. Storage cost exhibits a minimal impact on price dynamics, except within the initial quarters. A comparison with Figure 4 suggests that this marginal effect is primarily driven by the speculative channel, indicating that the bulk of the demand channel's impact stems from a direct demand shock. Any indirect effect of the demand shock through storage is likely limited. The effect of demand elasticity is straightforward: a less elastic demand corresponds to a more pronounced price decrease and a greater increase in stock levels.

The parameter α governs the sensitivity of the demand shock to economic activity and, as analyzed above, we calibrate its baseline value to $\alpha = 0.2$ to align with the commodity index dynamics of Bauer and Swanson (2023, Figure 8). Given the heterogeneous exposure of commodities to economic activity, the importance of the demand channel varies across different commodities. Modulating this parameter elucidates the relative significance of the demand channel compared to the speculative channel (case $\alpha = 0$).

5. Conclusion

This paper extends the classical competitive storage model to the setting where interest rates are time-varying. We developed a unified theory of how interest rates and other aggregate factors affect commodity prices. We proposed readily verifiable conditions under which a unique equilibrium solution exists and can be efficiently computed. These conditions have a natural interpretation in terms of the asymptotic yield on long-maturity risk-free assets. We also provided a sharp characterization of the analytical properties of the equilibrium objects and developed an efficient solution algorithm.

Within this framework, we investigated the dynamic causal effect of interest rates on commodity prices from a theoretical and quantitative perspective. On the theoretical side, we established conditions under which interest rates exert a negative effect on commodity prices or are negatively correlated with them. On the quantitative side, we applied our theory to explore the impact through the speculative and global demand channels. Impulse response analysis demonstrated a substantial and persistent negative effect of interest rates on commodity prices in most empirically relevant scenarios. Furthermore, the magnitude of this effect varies substantially depending on the prevailing market supply and interest rate regime.

Our quantitative application focuses on the speculative and the global demand channels. However, exploring (i) the impact of interest rates on commodity prices through various other channels, and (ii) the impact of more sophisticated stochastic discount factors (as found, for example, in Schorfheide et al., 2018) and risky returns are equally important. Although these topics lie beyond the scope of the current paper, the theory we develop lays a solid foundation for new work along these lines.

Appendix A. Proof of Section 2 Results

Here and in the remainder of the appendix, we let Φ be the probability transition matrix of $\{Z_t\}$. In particular, $\Phi(z, \hat{z})$ denotes the probability of transitioning from z to $\hat{z}$ in one step. Recall M_t defined in (6). We denote $\mathbb{E}_z := \mathbb{E}(\cdot \mid Z = z)$ and $\mathbb{E}_{\hat{z}} := \mathbb{E}(\cdot \mid \hat{Z} = \hat{z})$, and introduce the matrix L defined by

$$L(z, \hat{z}) := \Phi(z, \hat{z}) \, \mathbb{E}_{\hat{z}} \, m(\hat{z}, \hat{\varepsilon}). \tag{A.1}$$

Here L is expressed as a function on $\mathsf{Z} \times \mathsf{Z}$ but can be represented in traditional matrix notation by enumerating Z. Specifically, if $\mathsf{Z} = \{z_1, \ldots, z_N\}$, then $L = \Phi D$, where $D := \operatorname{diag}\{\mathbb{E}_{z_1} M, \ldots, \mathbb{E}_{z_N} M\}$.

For a square matrix A, let $s(A)$ denote its spectral radius. In other words, $s(A) := \max_{\alpha \in \Lambda} |\alpha|$, where Λ is the set of eigenvalues of A.

Lemma A.1. *Given L defined in (A.1), the asymptotic yield satisfies $\kappa(M) = -\ln s(L)$.*

Proof. By induction, we can show that, for any function $h : \mathsf{Z} \to \mathbb{R}$ and $n \in \mathbb{N}$,

$$L^n h(z) = \mathbb{E}_z \left(\prod_{t=1}^{n} M_t \right) h(Z_n), \tag{A.2}$$

where L^n is the n-th composition of the operator L with itself or, equivalently, the n-th power of the matrix L. By Theorem 9.1 of Krasnosel'skii et al. (2012) and the positivity of L, we have

$$s(L) = \lim_{n \to \infty} \|L^n h\|^{1/n}, \tag{A.3}$$

where h is any function on Z that takes positive values, and $\|\cdot\|$ is any norm on the set of real-valued functions defined on Z. Letting $h \equiv 1$ and $\|f\| := \mathbb{E}\,|f(Z_0)|$, we obtain

$$s(L) = \lim_{n \to \infty} \left(\mathbb{E} \left| \mathbb{E}_{Z_0} \prod_{t=1}^{n} M_t \right| \right)^{1/n} = \lim_{n \to \infty} \left(\mathbb{E} \prod_{t=1}^{n} M_t \right)^{1/n} = \lim_{n \to \infty} q_n^{1/n},$$

where the first and the last equalities are by definition, and the second equality is due to the Markov property. Since the log function is continuous, we then have $\ln s(L) = \lim_{n \to \infty} \ln q_n / n = -\kappa(M)$ and the claim follows. □

Corollary A.1. *Assumption 2.1 holds if and only if $s(L) < \mathrm{e}^{\delta}$.*

This follows directly from $\kappa(M) = -\ln s(L)$. Below we routinely use the alternative version $s(L) < \mathrm{e}^{\delta}$ for Assumption 2.1.

In the main text we imposed $p(b) < \infty$ to simplify analysis. Here and below, we relax this assumption. We assume instead $p(b) \leq \infty$, and prove that all the theoretical results in Sections 2–3 still hold in this generalized setup. To that end, we assume that

$$\mathbb{E}_z \max\{p(Y), 0\} < \infty \quad \text{for all } z \in \mathsf{Z}. \tag{A.4}$$

(This mild assumption confines the expected market value of commodity output to be finite and holds trivially in the setting of Section 2, where p is bounded above.) We then update the endogenous state space X and define it as

$$\mathsf{X} := \begin{cases} (b, \infty), & \text{if } p(b) = \infty, \\ [b, \infty), & \text{if } p(b) < \infty. \end{cases}$$

There is no loss of generality to truncate the endogenous state space when $p(b) = \infty$, because in this case, (A.4) implies that $Y_t > b$ almost surely, and thus $X_t > b$ with probability one for all t.

Let $p_0(x) := \max\{p(x), 0\}$ and let $\mathscr{C}$ be all continuous $f \colon \mathsf{S} \to \mathbb{R}$ such that f is decreasing in its first argument, $f(x,z) \geq p_0(x)$ for all $(x,z) \in \mathsf{S}$, and

$$\sup_{(x,z) \in \mathsf{S}} |f(x,z) - p_0(x)| < \infty.$$

Obviously, $\mathscr{C}$ reduces to the candidate space in Theorem 2.1 when the demand function p is bounded above, i.e., when $p(b) < \infty$. To compare pricing policies, we metrize $\mathscr{C}$ via

$$\rho(f,g) := \|f - g\| := \sup_{(x,z) \in \mathsf{S}} |f(x,z) - g(x,z)|.$$

Although f and g are not required to be bounded, one can show that ρ is a valid metric on $\mathscr{C}$ and that $(\mathscr{C}, \rho)$ is a complete metric space (see, e.g., Ma et al., 2020).

We aim to characterize the equilibrium pricing rule as the unique fixed point of the *equilibrium price operator* described as follows: For fixed $f \in \mathscr{C}$ and $(x,z) \in \mathsf{S}$, the value of Tf at (x,z) is defined as the $\xi \geq p_0(x)$ that solves

$$\xi = \psi(\xi, x, z) := \min\left\{\max\left\{\mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\mathrm{e}^{-\delta} I(\xi, x, z) + \hat{Y}, \hat{Z}\right) - k,\ p(x)\right\},\ p(b)\right\}, \quad \text{(A.5)}$$

where, considering free disposal,

$$I(\xi, x, z) := \begin{cases} x - p^{-1}(\xi), & \text{if } x < x_f^*(z) \\ x_f^*(z) - p^{-1}(\xi), & \text{if } x \geq x_f^*(z) \end{cases} \quad \text{(A.6)}$$

with

$$x_f^*(z) := \inf\left\{x \geq p^{-1}(0) \colon \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(0)] + \hat{Y}, \hat{Z}\right) - k = 0\right\}.$$

The domain of ψ is

$$\mathsf{G} := \{(\xi, x, z) \in \mathbb{R}_+ \times \mathsf{S} \colon \xi \in B(x)\}, \quad \text{(A.7)}$$

where $B(x)$ is defined for each x as

$$B(x) := \begin{cases} [p_0(x), \infty), & \text{if } p(b) = \infty, \\ [p_0(x), p(b)], & \text{if } p(b) < \infty. \end{cases} \quad \text{(A.8)}$$

Proposition A.1. *If $f \in \mathscr{C}$ and $(x,z) \in \mathsf{S}$, then $Tf(x,z)$ is uniquely defined.*

Proof. Fix $f \in \mathscr{C}$ and $(x,z) \in \mathsf{S}$. Since f is decreasing in its first argument and p^{-1} is decreasing (by the inverse function theorem), the map $\xi \mapsto \psi(\xi,x,z)$ is decreasing. Since the left-hand-side of equation (A.5) is strictly increasing in ξ, (A.5) can have at most one solution. Hence, uniqueness holds. Existence follows from the intermediate value theorem provided we can show that

(a) $\xi \mapsto \psi(\xi,x,z)$ is a continuous function,

(b) there exists $\xi \in B(x)$ such that $\xi \leq \psi(\xi,x,z)$, and

(c) there exists $\xi \in B(x)$ such that $\xi \geq \psi(\xi,x,z)$.

For part (a), it suffices to show that

$$g(\xi) := \mathbb{E}_z \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi,x,z), \hat{Z}\right)$$

is continuous on $B(x)$. To see this, fix $\xi \in B(x)$ and $\xi_n \to \xi$. Since $f \in \mathscr{C}$, there exists $D \in \mathbb{R}_+$ such that

$$0 \leq \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi_n,x,z), \hat{Z}\right) \leq \hat{M} f(\hat{Y},\hat{Z}) \leq \hat{M}\,[p_0(\hat{Y}) + D].$$

Since $\mathbb{E}_z \hat{M} p_0(\hat{Y}) = \mathbb{E}_z \left[\mathbb{E}_{\hat{z}} \hat{M}\, \mathbb{E}_{\hat{z}}\, p_0(\hat{Y})\right]$, the last term is integrable by (A.4). Hence, the dominated convergence theorem applies. From this fact and the continuity of f, p^{-1}, and I, we obtain $g(\xi_n) \to g(\xi)$. Hence, $\xi \mapsto \psi(\xi,x,z)$ is continuous.

Regarding part (b), consider $\xi = p_0(x)$. If $p(x) \geq 0$, then $\xi = p(x)$ and thus

$$\psi(\xi,x,z) \geq \min\{p(x), p(b)\} = p(x) = \xi.$$

If $p(x) < 0$, then $\xi = 0$. In this case, $I(\xi,x,z) = I(0,x,z) \leq x_f^*(z) - p^{-1}(0)$. The monotonicity of f and the definition of x_f^* then imply that

$$\mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\mathrm{e}^{-\delta} I(\xi,x,z) + \hat{Y}, \hat{Z}\right) - k \geq \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\mathrm{e}^{-\delta}[x_f^*(z) - p^{-1}(0)] + \hat{Y}, \hat{Z}\right) - k = 0.$$

By the definition of ψ,

$$\psi(\xi,x,z) \geq \min\{\max\{0, p(x)\}, p(b)\} = \min\{0, p(b)\} = 0 = \xi.$$

We have now verified part (b).

If $p(b) < \infty$, then part (c) holds by letting $\xi = p(b)$. If $p(b) = \infty$, then part (c) holds as ξ gets large since $\xi \mapsto \psi(\xi,x,z)$ is decreasing and bounded.

In summary, we have verified both existence and uniqueness. □

Proposition A.2. *$Tf \in \mathscr{C}$ for all $f \in \mathscr{C}$.*

Proof. Fix $f \in \mathscr{C}$ and define $g(\xi,x,z) := \mathbb{E}_z \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi,x,z), \hat{Z}\right)$.

First, we show that Tf is continuous. To this end, we first show that ψ in (A.5) is jointly continuous on the set G defined in (A.7). This will be true if g is jointly continuous on G. For any $\{(\xi_n,x_n,z_n)\}$ and (ξ,x,z) in G with $(\xi_n,x_n,z_n) \to (\xi,x,z)$, we need to show that $g(\xi_n,x_n,z_n) \to g(\xi,x,z)$. Define

$$h_1(\xi,x,z,\hat{Z},\hat{\varepsilon},\hat{\eta}),\, h_2(\xi,x,z,\hat{Z},\hat{\varepsilon},\hat{\eta}) := \hat{M} f(\hat{Y},\hat{Z}) \pm \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi,x,z), \hat{Z}\right),$$

where $\hat{M} := m(\hat{Z}, \hat{\varepsilon})$ and $\hat{Y} = y(\hat{Z}, \hat{\eta})$. Then h_1 and h_2 are continuous in $(\xi, x, z, \hat{Z})$ by the continuity of f, p^{-1}, and I, and non-negative by the monotonicity of f in its first argument.

Let π_ε and π_η denote respectively the probability measure of $\{\varepsilon_t\}$ and $\{\eta_t\}$. Fatou's lemma and Theorem 1.1 of Feinberg et al. (2014) imply that

$$\begin{aligned}
&\int\int \sum_{\hat{z}\in \mathsf{Z}} h_i(\xi, x, z, \hat{z}, \hat{\varepsilon}, \hat{\eta})\Phi(z, \hat{z})\pi_\varepsilon \,\mathrm{d}(\hat{\varepsilon})\pi_\eta \,\mathrm{d}(\hat{\eta}) \\
&\leq \int\int \liminf_{n\to\infty} \sum_{\hat{z}\in \mathsf{Z}} h_i(\xi_n, x_n, z_n, \hat{z}, \hat{\varepsilon}, \hat{\eta})\Phi(z_n, \hat{z})\pi_\varepsilon(\mathrm{d}\,\hat{\varepsilon})\pi_\eta(\mathrm{d}\,\hat{\eta}) \\
&\leq \liminf_{n\to\infty} \int\int \sum_{\hat{z}\in \mathsf{Z}} h_i(\xi_n, x_n, z_n, \hat{z}, \hat{\varepsilon}, \hat{\eta})\Phi(z_n, \hat{z})\pi_\varepsilon(\mathrm{d}\,\hat{\varepsilon})\pi_\eta(\mathrm{d}\,\hat{\eta}).
\end{aligned}$$

Since in addition $z \mapsto \mathbb{E}_z \hat{M} f(\hat{Y}, \hat{Z})$ is continuous, we have

$$\pm \mathbb{E}_z \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z),\, \hat{Z}\right) \leq \liminf_{n\to\infty} \left(\pm \mathbb{E}_{z_n} \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi_n, x_n, z_n),\, \hat{Z}\right)\right).$$

Then g is continuous, since the above inequality is equivalent to

$$\limsup_{n\to\infty} g(\xi_n, x_n, z_n) \leq g(\xi, x, z) \leq \liminf_{n\to\infty} g(\xi_n, x_n, z_n).$$

Hence, ψ is continuous on G, as was to be shown. Since $\xi \mapsto \psi(\xi, x, z)$ takes values in

$$\Gamma(x, z) := \left[p_0(x),\, \min\left\{p(b),\, p_0(x) + \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M}(p_0(\hat{Y}) + D)\right\}\right]$$

for some $D \in \mathbb{R}_+$, and the correspondence $(x, z) \mapsto \Gamma(x, z)$ is nonempty, compact-valued and continuous, Theorem B.1.4 of Stachurski (2009) implies that Tf is continuous on S.

Second, we show that Tf is decreasing in x. Suppose for some $z \in \mathsf{Z}$ and $x_1, x_2 \in \mathsf{X}$ with $x_1 < x_2$, we have $\xi_1 := Tf(x_1, z) < Tf(x_2, z) =: \xi_2$. Since f is decreasing in its first argument by assumption and I defined in (A.6) is increasing in ξ and x, ψ is decreasing in ξ and x. Then $\xi_2 > \xi_1 = \psi(\xi_1, x_1, z) \geq \psi(\xi_2, x_2, z) = \xi_2$, which is a contradiction.

Third, we show that $\sup_{(x,z)\in \mathsf{S}} |Tf(x, z) - p_0(x)| < \infty$. This obviously holds since

$$\begin{aligned}
&|Tf(x, z) - p_0(x)| = Tf(x, z) - p_0(x) \\
&\leq \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I\left(Tf(x, z), x, z\right), \hat{Z}\right) \leq \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M}[p_0(\hat{Y}) + D]
\end{aligned}$$

for all $(x, z) \in \mathsf{S}$ and some $D \in \mathbb{R}_+$, and the last term is finite by (A.4).

Finally, Proposition A.1 implies that $Tf(x, z) \in B(x)$ for all $(x, z) \in \mathsf{S}$. In conclusion, we have shown that $Tf(x, z) \in \mathscr{C}$. □

Lemma A.2. *T is order preserving on $\mathscr{C}$. That is, $Tf_1 \leq Tf_2$ for all $f_1, f_2 \in \mathscr{C}$ with $f_1 \leq f_2$.*

Proof. Let f_1, f_2 be functions in $\mathscr{C}$ with $f_1 \leq f_2$. Recall ψ defined in (A.5). With a slight abuse of notation, we define ψ_f such that $\psi_f(Tf(x, z), x, z) = Tf(x, z)$ for $f \in \{f_1, f_2\}$. Then $f_1 \leq f_2$ implies that $\psi_{f_1} \leq \psi_{f_2}$. Suppose to the contrary that there exits $(x, z) \in \mathsf{S}$ such that $\xi_1 := Tf_1(x, z) > Tf_2(x, z) = \xi_2$.

Since we have shown that $\xi \mapsto \psi(\xi, x, z)$ is decreasing for each $f \in \mathscr{C}$ and $(x,z) \in \mathsf{S}$, we have $\xi_1 = \psi_{f_1}(\xi_1, x, z) \leq \psi_{f_2}(\xi_1, x, z) \leq \psi_{f_2}(\xi_2, x, z) = \xi_2$, which is a contradiction. Therefore, T is order preserving. $\square$

Lemma A.3. *There exist $N \in \mathbb{N}$ and $\alpha \in (0,1)$ such that, for all $n \geq N$,*

$$\max_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} \mathrm{e}^{-\delta} M_t < \alpha^n.$$

Moreover, $D_1 := \sum_{t=0}^{\infty} \max_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{i=1}^{t} \mathrm{e}^{-\delta} M_i < \infty$.

Proof. The second inequality follows immediately from the first inequality. To verify the first inequality, note that letting $h \equiv 1$ and $\|f\| = \max_{z \in \mathsf{Z}} |f(z)|$ in (A.3) yields

$$s(L) = \lim_{n \to \infty} \left(\max_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} M_t \right)^{1/n}.$$

Since $\mathrm{e}^{-\delta} s(L) < 1$ by Corollary A.1, there exists $N \in \mathbb{N}$ and $\alpha < 1$ such that for all $n \geq N$,

$$\mathrm{e}^{-\delta} \left(\max_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} M_t \right)^{1/n} = \left(\max_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} \mathrm{e}^{-\delta} M_t \right)^{1/n} < \alpha.$$

Hence, the first inequality holds, and the proof is now complete. $\square$

To simplify notation, for given $\hat{Y}$, we denote

$$h(\xi, x, z) := \hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z) \quad \text{and} \quad g(\zeta, x) := \min \left\{ \max\{\zeta - k, p(x)\}, p(b) \right\}. \tag{A.9}$$

By definition, $\xi \mapsto h(\xi, x, z)$ and $\zeta \mapsto g(\zeta, x)$ are increasing given (x, z).

Lemma A.4. *For all $m \in \mathbb{N}$, $(x,z) \in \mathsf{S}$, and $\gamma \geq 0$, we have*

$$T^m (f + \gamma)(x, z) \leq T^m f(x, z) + \gamma \, \mathbb{E}_z \prod_{t=1}^{m} \mathrm{e}^{-\delta} M_t. \tag{A.10}$$

Proof. Fix $f \in \mathscr{C}$, $\gamma \geq 0$, and let $f_\gamma(x,z) := f(x,z) + \gamma$. By the definition of T,

$$\begin{aligned} T f_\gamma(x,z) &= g\left[\mathrm{e}^{-\delta} \, \mathbb{E}_z \, \hat{M} f_\gamma \left(h[T f_\gamma(x,z), x, z], \hat{Z} \right), x \right] \\ &\leq g\left[\mathrm{e}^{-\delta} \, \mathbb{E}_z \, \hat{M} f \left(h[T f_\gamma(x,z), x, z], \hat{Z} \right), x \right] + \gamma \, \mathbb{E}_z \, \mathrm{e}^{-\delta} \hat{M} \\ &\leq g\left[\mathrm{e}^{-\delta} \, \mathbb{E}_z \, \hat{M} f \left(h[T f(x,z), x, z], \hat{Z} \right), x \right] + \gamma \, \mathbb{E}_z \, \mathrm{e}^{-\delta} \hat{M}, \end{aligned}$$

where the second inequality is due to the fact that $f \leq f_\gamma$ and T is order preserving. Hence, $T(f+\gamma)(x,z) \leq Tf(x,z) + \gamma \, \mathbb{E}_z \, \mathrm{e}^{-\delta} \hat{M}$ and (A.10) holds for $m = 1$. Suppose (A.10) holds for arbitrary m. It remains to show that it holds for $m+1$. For $z \in \mathsf{Z}$, let

$\ell(z) := \gamma \, \mathbb{E}_z \prod_{t=1}^m \mathrm{e}^{-\delta} M_t$. By the induction hypothesis, Lemma A.2, and the Markov property,

$$
\begin{aligned}
T^{m+1} f_\gamma(x,z) &= g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}(T^m f_\gamma)\left(h[T^{m+1} f_\gamma(x,z), x, z], \hat{Z}\right), x\right] \\
&\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}(T^m f + \ell)\left(h[T^{m+1} f_\gamma(x,z), x, z], \hat{Z}\right), x\right] \\
&\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}(T^m f)\left(h[T^{m+1} f_\gamma(x,z), x, z], \hat{Z}\right), x\right] + \mathbb{E}_z\, \mathrm{e}^{-\delta} M_1 \ell(Z_1) \\
&\leq T^{m+1} f(x,z) + \gamma\, \mathbb{E}_z\, \mathrm{e}^{-\delta} M_1\, \mathbb{E}_{Z_1}\, \mathrm{e}^{-\delta} M_1 \cdots \mathrm{e}^{-\delta} M_m \\
&= T^{m+1} f(x,z) + \gamma\, \mathbb{E}_z \prod_{t=1}^{m+1} \mathrm{e}^{-\delta} M_t.
\end{aligned}
$$

Hence (A.10) holds by induction. □

Lemma A.5. *There exist $n \in \mathbb{N}$ and $\theta \in (0,1)$ such that*

$$T^n(f+\gamma) \leq T^n f + \theta\gamma \quad \textit{for all } f \in \mathscr{C} \textit{ and } \gamma \geq 0.$$

Proof. By the first part of Lemma A.3, there exist $n \in \mathbb{N}$ and $\alpha \in (0,1)$ such that

$$\mathbb{E}_z \prod_{t=1}^{n} \mathrm{e}^{-\delta} M_t < \alpha^n \quad \text{for all } z \in \mathsf{Z}.$$

Letting $\theta := \alpha^n$, we have $\theta < 1$. The stated claim then follows from Lemma A.4. □

Theorem A.1. *If Assumption 2.1 holds, then T is well defined on the function space $\mathscr{C}$, and there exists an $n \in \mathbb{N}$ such that T^n is a contraction mapping on $(\mathscr{C}, \rho)$. Moreover,*

(i) *T has a unique fixed point f^* in $\mathscr{C}$.*
(ii) *The fixed point f^* is the unique equilibrium pricing rule in $\mathscr{C}$.*
(iii) *For each f in $\mathscr{C}$, we have $\rho(T^k f, f^*)$ as $k \to \infty$.*

Proof. Proposition A.1 shows that T is a well-defined operator on $\mathscr{C}$. Since in addition (i) T is order preserving by Lemma A.2, (ii) $\mathscr{C}$ is closed under the addition of non-negative constants, and (iii) there exist $n \in \mathbb{N}$ and $\theta < 1$ such that $T^n(f+\gamma) \leq T^n f + \theta\gamma$ for all $f \in \mathscr{C}$ and $\gamma \geq 0$ by Lemma A.5, we have: T^n is a contraction mapping on $(\mathscr{C}, \rho)$ of modulus θ based on Blackwell (1965). Claims (i)–(iii) then follow from the Banach contraction mapping theorem and the definition of the equilibrium pricing rule. □

For each f in $\mathscr{C}$, we define

$$\bar{p}_f^0(z) := \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f(\hat{Y}, \hat{Z}) - k \quad \text{and} \quad \bar{p}_f(z) := \min\{\bar{p}_f^0(z), p(b)\}.$$

Lemma A.6. *For each f in $\mathscr{C}$, Tf satisfies*

(i) *$Tf(x,z) = p(x)$ if and only if $x \leq p^{-1}[\bar{p}_f(z)]$,*
(ii) *$Tf(x,z) > p_0(x)$ if and only if $p^{-1}[\bar{p}_f(z)] < x < x_f^*(z)$, and*
(iii) *$Tf(x,z) = 0$ if and only if $x \geq x_f^*(z)$.*

Proof. Regarding claim (i), suppose $Tf(x,z) = p(x)$. We show that $x \leq p^{-1}[\bar{p}_f(z)]$. Note that in this case, $x \leq p^{-1}(0) \leq x_f^*(z)$ since $p(x) = Tf(x,z) \geq 0$. Hence,

$$I\left[Tf(x,z), x, z\right] = x - p^{-1}[Tf(x,z)] = 0.$$

By the definition of T, we have

$$\begin{aligned} p(x) = Tf(x,z) &= \min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f(\hat{Y},\hat{Z}) - k, p(x)\right\}, p(b)\right\} \\ &\geq \min\left\{\bar{p}_f^0(z), p(b)\right\} = \bar{p}_f(z). \end{aligned}$$

Since p is decreasing, this implies $x \leq p^{-1}[\bar{p}_f(z)]$.

Next, we prove that $x \leq p^{-1}[\bar{p}_f(z)]$ implies $Tf(x,z) = p(x)$. If $\bar{p}_f^0(z) \geq p(b)$, then

$$\bar{p}_f(z) = p(b) \quad \Longrightarrow \quad x \leq p^{-1}[\bar{p}_f(z)] = p^{-1}[p(b)] = b.$$

Hence $x = b$. Then by definition $Tf(x,z) = \min\{\bar{p}_f^0(z), p(b)\} = p(b) = p(x)$.

If $\bar{p}_f^0(z) < p(b)$, then $\bar{p}_f(z) = \bar{p}_f^0(z)$. Since in addition

$$\bar{p}_f^0(z) \geq \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}p(\hat{Y}) - k > 0 \quad \text{and} \quad x \leq p^{-1}[\bar{p}_f(z)],$$

we have $x < p^{-1}(0) \leq x_f^*(z)$ in this case. Suppose to the contrary that $Tf(x,z) > p(x)$ for some $(x,z) \in \mathsf{S}$. Then by the definition of T,

$$p(x) < \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left[\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x,z)]\right) + \hat{Y}, \hat{Z}\right] - k.$$

The monotonicity of f in its first argument then implies that

$$p(x) < \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f(\hat{Y},\hat{Z}) - k = \bar{p}_f^0(z) = \bar{p}_f(z),$$

which is a contradiction. Claim (i) is now verified.

Note that claim (ii) follows immediately once claim (iii) is verified. To see that claim (iii) is true, suppose to the contrary that $x \geq x_f^*(x)$ and $Tf(x,z) > 0$ for some $(x,z) \in \mathsf{S}$. Then

$$I\left[Tf(x,z), x, z\right] = x_f^*(z) - p^{-1}[Tf(x,z)] > x_f^*(z) - p^{-1}(0).$$

By the definition of $x_f^*(z)$ and the monotonicity of f, this gives

$$\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}I\left[Tf(x,z), x, z\right] + \hat{Y}, \hat{Z}\right) - k \leq 0.$$

Using the definition of T, we obtain $0 < Tf(x,z) \leq \min\{\max\{0, p(x)\}, p(b)\} = 0$, which is a contradiction. Hence, $x \geq x_f^*(z)$ implies $Tf(x,z) = 0$.

Now suppose $Tf(x,z) = 0$. The definition of T implies that

$$\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}I(0,x,z) + \hat{Y}, \hat{Z}\right) - k \leq 0.$$

By the definition of $x_f^*(z)$, this gives $x \geq x_f^*(z)$. Claim (iii) is now verified. $\square$

Proof of Theorem 2.1. Theorem A.1 implies that there exists a unique equilibrium pricing rule f^* in $\mathscr{C}$. Claims (i)–(iii) follow immediately from Lemma A.6 since $\bar{p}(z) = \bar{p}_{f^*}(z)$ and f^* is the unique fixed point of T in $\mathscr{C}$.

To see that claim (iv) holds, suppose $f^*(x,z)$ is not strictly decreasing in x under the given conditions. Then by claims (i)–(iii), there exists $z \in \mathsf{Z}$ with $\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} < 1$ and a first interval $[x_0, x_1] \subset \left(p^{-1}[\bar{p}(z)], x^*(z)\right)$ such that $f^*(x,z) \equiv B$ on this interval for some constant $B > 0$. By the definition of T, for all $x \in [x_0, x_1]$,

$$B = f^*(x,z) = \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} f\left(\mathrm{e}^{-\delta} I\left[f^*(x,z), x, z\right] + \hat{Y}, \hat{Z}\right) - k.$$

Since the left-hand-side is a constant, $f\left(\mathrm{e}^{-\delta} I\left[f^*(x,z), x, z\right] + \hat{Y}, \hat{Z}\right) = B'(\hat{Z})$ for some constant $B'(\hat{Z})$. Moreover, $B'(\hat{Z}) \leq B$ since f is decreasing in x and $[x_0, x_1]$ is the first interval on which f is constant in x. Since in addition $\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} < 1$, we have $B \leq \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}B - k < B - k \leq B$, which is a contradiction. Hence claim (iv) must be true. □

Proof of Proposition 2.1. The continuity of i^* and claims (i)–(iii) follow from Theorem 2.1 and the definition of i^*. We next show that $i^*(x,z)$ is increasing in x. Since $i^*(x,z)$ is constant given z when $x \leq p^{-1}[\bar{p}(z)]$ and when $x \geq x^*(z)$ by claim (i) and claim (iii), it remains to show that $i^*(x,z)$ is increasing in x when $p^{-1}[\bar{p}(z)] < x < x^*(z)$. In this case $i^*(x,z) = x - p^{-1}[f^*(x,z)]$. Suppose to the contrary that there exist $z \in \mathsf{Z}$ and $x_1, x_2 \in \left(p^{-1}[\bar{p}(z)], x^*(z)\right)$ such that $x_1 < x_2$ and $i^*(x_1, z) > i^*(x_2, z)$. Then by definition,

$$x_1 - p^{-1}[f^*(x_1,z)] > x_2 - p^{-1}[f^*(x_2,z)].$$

Since $x_1 < x_2$, this gives $p^{-1}[f^*(x_2,z)] > p^{-1}[f^*(x_1,z)]$. But by (ii) of Theorem 2.1 and the definition of T, we obtain

$$\begin{aligned} f^*(x_1,z) &= \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_1,z) + \hat{Y}, \hat{Z}\right) - k \\ &\leq \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_2,z) + \hat{Y}, \hat{Z}\right) - k \leq f^*(x_2,z), \end{aligned}$$

which implies $p^{-1}[f^*(x_1,z)] \geq p^{-1}[f^*(x_2,z)]$. This is a contradiction. Hence, it is true that $i^*(x,z)$ is increasing in x.

It remains to verify claim (iv). Pick any $z \in \mathsf{Z}$ and $x_1, x_2 \in \left(p^{-1}[\bar{p}(z)], x^*(z)\right)$ with $x_1 < x_2$. By claim (iv) of Theorem 2.1, we have $f^*(x_1,z) > f^*(x_2,z)$. Using the definition of T and claim (ii) of Theorem 2.1 again, we have

$$\mathbb{E}_z\,\hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_1,z) + \hat{Y}, \hat{Z}\right) > \mathbb{E}_z\,\hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_2,z) + \hat{Y}, \hat{Z}\right).$$

The monotonicity of f^* then gives $i^*(x_1,z) < i^*(x_2,z)$. Hence claim (iv) holds. □

Appendix B. Proof of Section 3 Results

A Markov chain $\{Z_t\}$ with transition matrix F is called *monotone* if

$$\int h(\hat{z})\,\mathrm{d}\,F(z_1, \hat{z}) \leq \int h(\hat{z})\,\mathrm{d}\,F(z_2, \hat{z})$$

whenever $z_1 \leq z_2$ and $h : \mathsf{Z} \to \mathbb{R}$ is bounded and increasing. In proofs, when stating equality or inequality conditions between different random variables, we understand them as holding with probability one.

Proof of Proposition 3.1. Let $\mathscr{C}_1$ be the elements in $\mathscr{C}$ such that $z \mapsto f(x,z)$ is decreasing for all x. Obviously, $\mathscr{C}_1$ is a closed subset of $\mathscr{C}$. Therefore, to show that $z \mapsto f^*(x,z)$ is decreasing for all x, it suffices to verify $T\mathscr{C}_1 \subset \mathscr{C}_1$.

Fix $f \in \mathscr{C}_1$ and $z_1, z_2 \in \mathsf{Z}$ with $z_1 \leq z_2$. Suppose there exists an x such that

$$\xi_1 := Tf(x, z_1) < Tf(x, z_2) =: \xi_2. \tag{B.1}$$

Note that f is a decreasing function since $f \in \mathscr{C}_1$. Moreover, by assumption $m(z,\varepsilon) = 1/r(z,\varepsilon)$ is decreasing in z, $y(z,\eta)$ is increasing in z, and Φ is monotone. Therefore, for all $\xi \in B(x)$, we have

$$\mathbb{E}_{z_1} \hat{M} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi)] + \hat{Y}, \hat{Z}\right) \geq \mathbb{E}_{z_2} \hat{M} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi)] + \hat{Y}, \hat{Z}\right). \tag{B.2}$$

In particular, by the definition of x_f^*, we have $x_f^*(z_2) \leq x_f^*(z_1)$. If $x < x_f^*(z_2)$, then

$$I(\xi_1, x, z_1) = x - p^{-1}(\xi_1) \leq x - p^{-1}(\xi_2) = I(\xi_2, x, z_2).$$

Recall ψ defined in (A.5). The above inequality and (B.2) imply that

$$\xi_1 = \psi(\xi_1, x, z_1) \geq \psi(\xi_1, x, z_2) \geq \psi(\xi_2, x, z_2) = \xi_2.$$

If $x \geq x_f^*(z_2)$, then we also have $\xi_1 \geq \xi_2$ since $\xi_2 = 0$ and $\xi_1 \geq 0$. In either case, this is contradicted with (B.1). Therefore, we have shown that $z \mapsto Tf(x,z)$ is decreasing for all x and $T\mathscr{C}_1 \subset \mathscr{C}_1$. It then follows that $z \mapsto f^*(x,z)$ is decreasing for all x.

To see that $i^*(x,z)$ is decreasing in z, pick any $z_1, z_2 \in \mathsf{Z}$ with $z_1 \leq z_2$. By the definition of $x^*(z)$ and the monotonicity of $f^*(x,z)$ in z, we have

$$0 = f^*(x^*(z_1), z_1) \geq f^*(x^*(z_1), z_2)$$

and thus $x^*(z_1) \geq x^*(z_2)$. The definition of i^* and the monotonicity of p^{-1} and f^* then implies that

$$\begin{aligned} i^*(x, z_1) &= \min\{x, x^*(z_1)\} - p^{-1}[f^*(x, z_1)] \\ &\geq \min\{x, x^*(z_2)\} - p^{-1}[f^*(x, z_2)] = i^*(x, z_2). \end{aligned}$$

Hence $z \mapsto i^*(x,z)$ is decreasing for all x.

Finally, note that $\hat{Z} \mapsto \hat{M} f^*(\hat{Y}, \hat{Z}) = m(\hat{Z}, \hat{\varepsilon}) f^*(y(\hat{Z}, \hat{\eta}), \hat{Z})$ is decreasing because f^* is decreasing, y is increasing in z, and m is decreasing in z. Since in addition $\{Z_t\}$ is monotone, it follows immediately by definition that $z \mapsto \mathbb{E}_z \hat{M} f^*(\hat{Y}, \hat{Z})$ is decreasing. Hence $\bar{p}$ is decreasing by definition. □

Next, we discuss the correlation between commodity price and stochastic discount factor. To state the result, we suppose $Z_t = (Z_{1t}, \dots, Z_{nt})$ takes values in $\mathbb{R}^n$. The following is a simple corollary of the key result of Fortuin et al. (1971).

Lemma B.1 (Fortuin–Kasteleyn–Ginibre). *If f, g are decreasing integrable functions on $\mathbb{R}^n$ and $W = (W_1, \cdots, W_n)$ is a random vector on $\mathbb{R}^n$ such that $\{W_1, \cdots, W_n\}$ are independent, then $\mathbb{E} f(W) \mathbb{E} g(W) \leq \mathbb{E} f(W) g(W)$.*

Lemma B.1 implies that if f is decreasing and g is nondecreasing (so that $-g$ is decreasing), then we have $\mathbb{E} f(W) \mathbb{E} g(W) \geq \mathbb{E} f(W) g(W)$.

Proposition B.1. *If $m(z, \varepsilon)$ is decreasing in z, $y(z, \eta)$ is nondecreasing in z, Φ is monotone, and $\{Z_{1t}, \ldots, Z_{nt}\}$ are independent for each fixed t, then $\operatorname{Cov}_{t-1}(P_t, M_t) \geq 0$ for all $t \in \mathbb{N}$.*

Proof. The equilibrium path is $X_t = \mathrm{e}^{-\delta} i^*(X_{t-1}, Z_{t-1}) + y(Z_t, \eta_t)$ where

$$i^*(X_{t-1}, Z_{t-1}) = \min\{X_{t-1}, x^*(Z_{t-1})\} - p^{-1}\left[f^*(X_{t-1}, Z_{t-1})\right].$$

Note that X_t is a nondecreasing function of Z_t since $z \mapsto y(z, \eta)$ is nondecreasing for all η. Moreover, the proof of Proposition 3.1 implies that f^* is a decreasing function under the assumptions of the current proposition. Hence, $Z_t \mapsto f^*(X_t, Z_t)$ is decreasing. Since in addition $z \mapsto m(z, \varepsilon)$ is decreasing for all ε and $\{Z_{1t}, \ldots, Z_{nt}\}$ are independent, applying Lemma B.1 (taking $W = Z_t$) yields

$$\begin{aligned}
&\mathbb{E}\left[f^*(X_t, Z_t) m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t\right] \\
&\geq \mathbb{E}\left[f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t\right] \mathbb{E}\left[m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t\right] \\
&= \mathbb{E}\left[f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}, \eta_t\right] \mathbb{E}\left[m(Z_t, \varepsilon_t) \mid Z_{t-1}, \varepsilon_t\right].
\end{aligned}$$

Using this result, it follows that

$$\begin{aligned}
&\mathbb{E}\left(P_t M_t \mid X_{t-1}, Z_{t-1}\right) = \mathbb{E}\left[f^*(X_t, Z_t) m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}\right] \\
&= \mathbb{E}\left(\mathbb{E}\left[f^*(X_t, Z_t) m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t\right] \mid X_{t-1}, Z_{t-1}\right) \\
&\geq \mathbb{E}\left\{\mathbb{E}\left[f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}, \eta_t\right] \mathbb{E}\left[m(Z_t, \varepsilon_t) \mid Z_{t-1}, \varepsilon_t\right] \mid X_{t-1}, Z_{t-1}\right\} \\
&= \mathbb{E}\left[f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}\right] \mathbb{E}\left[m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}\right] \\
&= \mathbb{E}\left(P_t \mid X_{t-1}, Z_{t-1}\right) \mathbb{E}\left(M_t \mid X_{t-1}, Z_{t-1}\right),
\end{aligned}$$

where the second-to-last equality holds because η_t is independent of ε_t. Hence,

$$\begin{aligned}
&\operatorname{Cov}_{t-1}(P_t, M_t) = \operatorname{Cov}(P_t, M_t \mid X_{t-1}, Z_{t-1}) \\
&= \mathbb{E}\left(P_t, M_t \mid X_{t-1}, Z_{t-1}\right) - \mathbb{E}\left(P_t \mid X_{t-1}, Z_{t-1}\right) \mathbb{E}\left(M_t \mid X_{t-1}, Z_{t-1}\right) \geq 0,
\end{aligned}$$

as was to be shown. □

Proof of Proposition 3.2. Since $R_t = 1/M_t$, applying Lemma B.1 again and working through similar steps to the proof of Proposition B.1, we can show that $\operatorname{Cov}_{t-1}(P_t, R_t) \leq 0$ for all t. The details are omitted. □

Proof of Proposition 3.3. Let T_1 and T_2 be respectively the equilibrium price operators corresponding to $\{R_t^1\}$ and $\{R_t^2\}$. It suffices to show that $T_1 f \leq T_2 f$ for all $f \in \mathscr{C}$. To see this, we adopt an induction argument. Suppose $T_1^k f \leq T_2^k f$. Then by the order preserving property of the equilibrium price operator and the initial argument $T_1 f \leq T_2 f$ for all f in

$\mathscr{C}$, we have $T_1^{k+1} f = T_1(T_1^k f) \leq T_1(T_2^k f) \leq T_2(T_2^k f) = T_2^{k+1} f$. Hence, $T_1^k f \leq T_2^k f$ for all $k \in \mathbb{N}$ and $f \in \mathscr{C}$. Letting $k \to \infty$ then yields $f_1^* \leq f_2^*$.

We now show that $T_1 f \leq T_2 f$ for all $f \in \mathscr{C}$. Suppose there exists $(x, z) \in \mathsf{S}$ such that $\xi_1 := T_1 f(x, z) > T_2 f(x, z) =: \xi_2$. Let $M_t^i = 1/R_t^i$ for $i = 1, 2$. Since $R_t^1 \geq R_t^2$, we have $M_t^1 \leq M_t^2$. Letting $\hat{M}_i$ be the next-period discount factor of economy i, the monotonicity of g and h defined in (A.9) then implies that

$$\begin{aligned}\xi_1 &= g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}_1 f\left(h(\xi_1, x), \hat{Z}\right), x\right] \\ &\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}_2 f\left(h(\xi_1, x), \hat{Z}\right), x\right] \leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}_2 f\left(h(\xi_2, x), \hat{Z}\right), x\right] = \xi_2,\end{aligned}$$

which is a contradiction. Therefore, $T_1 f \leq T_2 f$ and all the stated claims hold. □

Proof of Proposition 3.4. In this case, the exogenous state is $Z_t = R_t$, which is a monotone Markov process, and $r(z, \varepsilon) \equiv z$ is strictly increasing in z. Since $R_t^2 \leq R_t^1$, y is nondecreasing in R, and both economies share the same innovation process $\{\eta_t\}$, we have $Y_t^2 \leq Y_t^1$. Since in addition $X_{t-1}^2 \leq X_{t-1}^1$ and $R_{t-1}^1 \leq R_{t-1}^2$, Propositions 2.1 and 3.2 and the law of motion of the state process imply that

$$X_t^2 = \mathrm{e}^{-\delta} i^*(X_{t-1}^2, R_{t-1}^2) + Y_t^2 \leq \mathrm{e}^{-\delta} i^*(X_{t-1}^1, R_{t-1}^1) + Y_t^1 = X_t^1$$

with probability one. Applying Proposition 3.2 again and the monotonicity of f^* with respect to the endogenous state yields

$$P_t^1 = f^*(X_t^1, R_t^1) \leq f^*(X_t^1, R_t^2) \leq f^*(X_t^2, R_t^2) = P_t^2$$

with probability one. Hence the statement is verified. □

Proof of Proposition 3.5. We have seen that, in this case, the exogenous state is $Z_t = R_t$, which is a monotone Markov process, and $r(z, \varepsilon) \equiv z$ is strictly increasing in z. Since $R_t \leq R_{t+1}$ and y is nondecreasing, we have $Y_t \leq Y_{t+1}$. Since in addition $X_{t-1} \leq X_t$ and $R_{t-1} \geq R_t$, applying Propositions 2.1 and 3.2, the law of motion of the state process indicates that

$$X_t = \mathrm{e}^{-\delta} i^*(X_{t-1}, R_{t-1}) + Y_t \leq \mathrm{e}^{-\delta} i^*(X_t, R_t) + Y_{t+1} = X_{t+1}$$

with probability one. Then Proposition 3.2 and the monotonicity of f^* with respect to the endogenous state imply that

$$P_t = f^*(X_t, R_t) \geq f^*(X_t, R_{t+1}) \geq f^*(X_{t+1}, R_{t+1}) = P_{t+1}$$

with probability one. Hence the stated claim holds. □

Appendix C. Positive Correlation

Here we provide examples showing that Proposition 3.2 does not hold in general if Z_t is positively or negatively correlated across dimensions. We begin with the following.

Proposition C.1. *If Assumption 2.1 holds and the inverse demand function is $p(x) = a + dx$ with $a > 0$ and $d < 0$, then the equilibrium pricing rule $f^*(x, z)$ is convex in x.*

Proof. Let $\mathscr{C}_2$ be the elements in $\mathscr{C}$ such that $x \mapsto f(x,z)$ is convex for all $z \in \mathsf{Z}$. Then $\mathscr{C}_2$ is a closed subset of $\mathscr{C}$. Hence it suffices to show that $T\mathscr{C}_2 \subset \mathscr{C}_2$. Fix $f \in \mathscr{C}_2$, since $Tf \in \mathscr{C}$ by Proposition A.2, it remains to show that $Tf(x,z)$ is convex in x. Since $Tf(x,z)$ is decreasing in x and, by Lemma A.6, $Tf(x,z)$ is linear in x when $x \leq p^{-1}[\bar{p}(z)]$ or $x \geq x_f^*(z)$, it suffices to show that $Tf(x,z)$ is convex in x on $B_0(z) := (p^{-1}[\bar{p}(z)], x_f^*(z))$. In this case,

$$Tf(x,z) = \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x,z)]\right) + \hat{Y}, \hat{Z}\right) - k.$$

Suppose to the contrary that $Tf(x,z)$ is not convex, then there exist $z \in \mathsf{Z}$, $x_1, x_2 \in B_0(z)$, and $\alpha \in [0,1]$ such that, letting $x_0 := \alpha x_1 + (1-\alpha)x_2$,

$$\begin{aligned}
&\alpha Tf(x_1,z) + (1-\alpha)Tf(x_2,z) < Tf(x_0,z) \\
&= \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x_0,z)]\right) + \hat{Y}, \hat{Z}\right) - k \\
&\leq \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}\left[\alpha Tf(x_1,z) + (1-\alpha)Tf(x_2,z)\right]\right) + \hat{Y}, \hat{Z}\right) - k \\
&\leq \alpha Tf(x_1,z) + (1-\alpha)Tf(x_2,z),
\end{aligned}$$

where the last inequality is by convexity of $f(x,z)$ in x and the linearity of $p(x)$. This is a contradiction. Hence $Tf(x,z)$ is convex in x on $B_0(z)$ and the stated claim holds. □

Suppose $R_t = 0.98$ with probability 0.5 and $R_t = 1.02$ with probability 0.5. If $R_t = 0.98$, then $Y_t = y_0$ with probability one, and if $R_t = 1.02$, then $Y_t = y_1$ with probability φ and $Y_t = y_2$ with probability $1-\varphi$. This is a special case of our framework. In particular,

$$\begin{gathered}
\varepsilon_t = \eta_t = 0, \quad Z_t = (Z_{1t}, Z_{2t}) = (R_t, Y_t), \\
r(Z_t, \varepsilon_t) = r(R_t, Y_t, \varepsilon_t) = R_t \quad \text{and} \quad y(Z_t, \eta_t) = y(R_t, Y_t, \eta_t) = Y_t.
\end{gathered}$$

Note that $\{Z_t\}$ is IID. Hence, it is naturally monotone and the equilibrium pricing rule is not a function of Z_t. Since in addition $r(z,\varepsilon)$ and $y(z,\eta)$ are increasing in z, the assumptions of Proposition 3.1 hold. However, because Z_{1t} and Z_{2t} (i.e., R_t and Y_t) are correlated, the assumptions of Proposition 3.2 are violated.

Some simple algebra shows that $\mathbb{E}\, R_t = 1$,

$$\mathbb{E}\, Y_t = \frac{y_0}{2} + \frac{\varphi y_1}{2} + \frac{(1-\varphi)y_2}{2} \quad \text{and} \quad \mathbb{E}\, R_t Y_t = \frac{0.98 y_0}{2} + \frac{1.02\varphi y_1}{2} + \frac{1.02(1-\varphi)y_2}{2}.$$

Hence $\mathrm{Cov}_{t-1}(R_t, Y_t) = \mathbb{E}\, R_t Y_t - \mathbb{E}\, R_t\, \mathbb{E}\, Y_t = -0.01 y_0 + 0.01\varphi y_1 + 0.01(1-\varphi)y_2$. Choose δ such that $\delta > \log \mathbb{E}(1/R_t) \approx 0.0004$. Then $\beta := \mathrm{e}^{-\delta}\, \mathbb{E}(1/R_t) < 1$ and the following result holds under the current setup.

Lemma C.1. *Either (i) $P_t = 0$ for all t or (ii) $I_t = 0$ in finite time with probability one. If the per-unit storage cost $k > 0$, then (ii) holds.*

Proof. Suppose (ii) does not hold, then $I_t > 0$ for all t with positive probability, and the equilibrium price path satisfies

$$P_0 \leq \mathrm{e}^{-\delta t}\,\mathbb{E}\left(\prod_{i=0}^{t}\frac{1}{R_i}\right)P_t - \left(\sum_{i=0}^{t-1}\mathrm{e}^{-\delta i}\right)k \quad \text{for all } t. \tag{C.1}$$

Note that $\{P_t\}$ is bounded with probability one since $f^* \in \mathscr{C}$ implies that, for some $L_0 < \infty$, we have $P_t = f^*(X_t) \leq f^*(Y_t) \leq f^*(\underline{y}) \leq p_0(\underline{y}) + L_0 =: L_1 < \infty$ with probability one, where $\underline{y} := \min\{y_0, y_1, y_2\}$. If in addition (i) does not hold, we may assume $P_0 > 0$ without loss of generality. In this case, (C.1) implies that, when t is sufficiently large, we have $0 < P_0 \leq \beta^t L_1 < P_0$ with positive probability, which is a contradiction. Hence either (i) or (ii) holds. If, on the other hand, $k > 0$ and (ii) does not hold, then for sufficiently large t, (C.1) implies that $P_0 < P_0$ with positive probability for all $P_0 \geq 0$, which is also a contradiction. Hence (ii) holds and the second claim is also verified. ☐

Consider a linear inverse demand function p as in Proposition C.1. If $P_t = 0$ for all t, then $\mathrm{Cov}(R_t, P_t) = \mathrm{Cov}_{t-1}(R_t, P_t) = 0$ for all t. Otherwise, $I_{t-1} = 0$ for some finite t, in which case $X_t = Y_t$, $P_t = f^*(Y_t)$, and thus

$$\begin{aligned}\mathrm{Cov}_{t-1}(R_t, P_t) &= \mathbb{E}_{t-1} R_t P_t - \mathbb{E} R_t\ \mathbb{E}_{t-1} P_t = \mathbb{E} R_t f^*(Y_t) - \mathbb{E} R_t\ \mathbb{E} f^*(Y_t) \\ &= -0.01 f^*(y_0) + 0.01\varphi f^*(y_1) + 0.01(1-\varphi) f^*(y_2).\end{aligned}$$

If $y_1, y_2 < y_0$, then $\mathrm{Cov}_{t-1}(R_t, Y_t) < 0$ and $\mathrm{Cov}_{t-1}(R_t, P_t) > 0$ based on the monotonicity of f^*. If on the other hand y_0, y_1 and y_2 satisfy[32] $y_1 < p^{-1}(\bar{p}) < y_0 < y_2$, then since f^* is convex by Proposition C.1, and $f^*(x) > p(x)$ whenever $x > p^{-1}(\bar{p})$ by Theorem 2.1,

$$\frac{f^*(y_0) - f^*(y_2)}{f^*(y_1) - f^*(y_2)} < \frac{y_2 - y_0}{y_2 - y_1}.$$

Hence φ can be chosen such that $y_2 - y_0 > \varphi(y_2 - y_1)$ and $f^*(y_0) - f^*(y_2) < \varphi[f^*(y_1) - f^*(y_2)]$. In particular, the above inequalities respectively imply that

$$\mathrm{Cov}_{t-1}(R_t, Y_t) > 0 \quad \text{and} \quad \mathrm{Cov}_{t-1}(R_t, P_t) > 0.$$

References

Alquist, Ron, Saroj Bhattarai, and Olivier Coibion, "Commodity-price comovement and global economic activity," *Journal of Monetary Economics*, 2020, *112*, 41–56.

Anzuini, Alessio, Marco J. Lombardi, and Patrizio Pagano, "The impact of monetary policy shocks on commodity prices," *International Journal of Central Banking*, 2013, *9* (3), 125–150.

Arseneau, David M. and Sylvain Leduc, "Commodity price movements in a general equilibrium model of storage," *IMF Economic Review*, 2013, *61* (1), 199–224.

[32] Since $\mathrm{e}^{-\delta}\,\mathbb{E}(1/R_t) < 1$, we have $\bar{p} = \min\{\mathrm{e}^{-\delta}\,\mathbb{E} f^*(\hat{Y})/\hat{R} - k, p(b)\} < p(b)$. Thus $p^{-1}(\bar{p}) > b$ and this choice of y_0, y_1, y_2 is feasible.

Baker, Steven D, "The financialization of storable commodities," *Management Science*, 2021, *67* (1), 471–499.

Barlevy, Gadi, "Rethinking Theoretical Models of Bubbles: Reflections Inspired by the Financial Crisis and Allen and Gorton's Paper "Churning Bubbles"," in Douglas D. Evanoff, George G. Kaufman, and A. G. Malliaris, eds., *New Perspectives on Asset Price Bubbles*, Oxford University Press, March 2012, chapter 3, pp. 41–62.

Basak, Suleyman and Anna Pavlova, "A model of financialization of commodities," *The Journal of Finance*, 2016, *71* (4), 1511–1556.

Bauer, Michael D. and Eric T. Swanson, "A Reassessment of Monetary Policy Surprises and High-Frequency Identification," *NBER Macroeconomics Annual*, 2023, *37*, 87–155.

Bernanke, Ben S, Jean Boivin, and Piotr Eliasz, "Measuring the effects of monetary policy: a factor-augmented vector autoregressive (FAVAR) approach," *The Quarterly Journal of Economics*, 2005, *120* (1), 387–422.

Blackwell, David, "Discounted dynamic programming," *Annals of Mathematical Statistics*, 1965, *36* (1), 226–235.

Bobenrieth, Eugenio S.A., Juan R.A. Bobenrieth, Ernesto A. Guerra, Brian D. Wright, and Di Zeng, "Putting the empirical commodity storage model back on track: crucial implications of a "negligible" trend," *American Journal of Agricultural Economics*, 2021, *103* (3), 1034–1057.

Bodart, Vincent, François Courtoy, and Erica Perego, "World Interest Rates and Macroeconomic Adjustments in Developing Commodity Producing Countries," Working Paper 2021-01, CEPII 2021.

Byrne, Joseph P, Giorgio Fazio, and Norbert Fiess, "Primary commodity prices: Co-movements, common factors and fundamentals," *Journal of Development Economics*, 2013, *101*, 16–26.

Cafiero, Carlo, Eugenio S.A. Bobenrieth, Juan R.A. Bobenrieth, and Brian D. Wright, "The empirical relevance of the competitive storage model," *Journal of Econometrics*, 2011, *162* (1), 44–54.

______, ______, ______, and ______, "Maximum likelihood estimation of the standard commodity storage model: Evidence from sugar prices," *American Journal of Agricultural Economics*, 2015, *97* (1), 122–136.

Carroll, Christopher D., "The method of endogenous gridpoints for solving dynamic stochastic optimization problems," *Economics Letters*, 2006, *91* (3), 312–320.

Casassus, Jaime and Pierre Collin-Dufresne, "Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates," *The Journal of Finance*, 2005, *60* (5), 2283–2331.

Chambers, Marcus J. and Roy E. Bailey, "A theory of commodity price fluctuations," *Journal of Political Economy*, 1996, *104* (5), 924–957.

Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans, "Monetary policy shocks: What have we learned and to what end?," in John B. Taylor and Michael Woodford, eds., *Handbook of Macroeconomics*, Vol. 1, Part A, Elsevier, 1999, chapter 2, pp. 65–148.

Cody, Brian J and Leonard O Mills, "The role of commodity prices in formulating monetary policy," *The Review of Economics and Statistics*, 1991, *73* (2), 358–365.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, "The relation between forward prices and futures prices," *Journal of Financial Economics*, 1981, *9* (4), 321–346.

Deaton, Angus and Guy Laroque, "On the behavior of commodity prices," *The Review of Economic Studies*, 1992, *59* (1), 1–23.

______ **and** ______, "Competitive storage and commodity price dynamics," *Journal of Political Economy*, 1996, *104* (5), 896–923.

Eberhardt, Markus and Andrea F Presbitero, "Commodity prices and banking crises," *Journal of International Economics*, 2021, *131*, 103474.

Fama, Eugene F. and Kenneth R. French, "Commodity futures prices: Some evidence on forecast power, premiums, and the theory of storage," *The Journal of Business*, 1987, *60* (1), 55–73.

Feinberg, Eugene A., Pavlo O. Kasyanov, and Nina V. Zadoianchuk, "Fatou's lemma for weakly converging probabilities," *Theory of Probability & Its Applications*, 2014, *58* (4), 683–689.

Fortuin, C. M., P. W. Kasteleyn, and J. Ginibre, "Correlation inequalities on some partially ordered sets," *Communications in Mathematical Physics*, 1971, *22* (2), 89–103.

Frankel, Jeffrey A., "Expectations and commodity price dynamics: The overshooting model," *American Journal of Agricultural Economics*, 1986, *68* (2), 344–348.

______, "The effect of monetary policy on real commodity prices," in John Y. Campbell, ed., *Asset Prices and Monetary Policy*, National Bureau of Economic Research, 2008, chapter 7, pp. 291–333.

______, "An explanation for soaring commodity prices," *VoxEU*, 2008.

______, "Monetary policy and commodity prices," *VoxEU*, 2008.

______, "Effects of speculation and interest rates in a "carry trade" model of commodity prices," *Journal of International Money and Finance*, 2014, *42*, 88–112.

______, "Rising US Real Interest Rates Imply Falling Commodity Prices," blog post, Harvard Kennedy School, Belfer Center for Science and International Affairs 2018.

______ **and Gikas A. Hardouvelis**, "Commodity prices, money surprises and Fed credibility," *Journal of Money, Credit, and Banking*, 1985, *17* (4), 425–438.

Gardner, Bruce L., "Futures Prices in Supply Analysis," *American Journal of Agricultural Economics*, 1976, *58* (1), 81–84.

Gibson, Rajna and Eduardo S. Schwartz, "Stochastic Convenience Yield and the Pricing of Oil Contingent Claims," *The Journal of Finance*, 1990, *45* (3), 959–976.

Goldstein, Itay and Liyan Yang, "Commodity financialization and information transmission," *The Journal of Finance*, 2022, *77* (5), 2613–2667.

Gospodinov, Nikolay and Serena Ng, "Commodity prices, convenience yields, and inflation," *Review of Economics and Statistics*, 2013, *95* (1), 206–219.

Gouel, Christophe and Nicolas Legrand, "Estimating the Competitive Storage Model with Trending Commodity Prices," *Journal of Applied Econometrics*, June 2017, *32* (4), 744–763.

______ **and** ______, "The role of storage in commodity markets: Indirect inference based on grains data," Working Paper 2022-04, CEPII 2022.

Gruber, Joseph W. and Robert J. Vigfusson, "Interest rates and the volatility and correlation of commodity prices," *Macroeconomic Dynamics*, 2018, *22* (3), 600–619.

Guerron-Quintana, Pablo A, Tomohiro Hirano, and Ryo Jinnai, "Bubbles, crashes, and economic growth: Theory and evidence," *American Economic Journal: Macroeconomics*, 2023, *15* (2), 333–371.

Harvey, David I., Neil M. Kellard, Jakob B. Madsen, and Mark E. Wohar, "Long-run commodity prices, economic growth, and interest rates: 17th century to the present day," *World Development*, 2017, *89*, 57–70.

Hirano, Tomohiro and Alexis Akira Toda, "Bubble economics," *Journal of Mathematical Economics*, 2024, *111*, 102944.

Kilian, Lutz, "Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market," *American Economic Review*, 2009, *99* (3), 1053–69.

______ **and Xiaoqing Zhou**, "Oil prices, exchange rates and interest rates," *Journal of International Money and Finance*, 2022, *126*, 102679.

Koop, Gary, M. Hashem Pesaran, and Simon M. Potter, "Impulse response analysis in nonlinear multivariate models," *Journal of Econometrics*, 1996, *74* (1), 119–147.

Krasnosel'skii, M. A., G. M. Vainikko, R.P. Zabreyko, Y. B. Ruticki, and V. V. Stet'senko, *Approximate Solution of Operator Equations*, Springer Netherlands, 2012.

Ma, Qingyin, John Stachurski, and Alexis Akira Toda, "The income fluctuation problem and the evolution of wealth," *Journal of Economic Theory*, 2020, *187*, 105003.

Newbery, David M. G. and Joseph E. Stiglitz, "Optimal commodity stock-piling rules," *Oxford Economic Papers*, 1982, pp. 403–427.

Peersman, Gert, "International food commodity prices and missing (dis)inflation in the euro area," *Review of Economics and Statistics*, 2022, *104* (1), 85–100.

Plantin, Guillaume, "Asset bubbles and inflation as competing monetary phenomena," *Journal of Economic Theory*, 2023, *212*, 105711.

Ramey, Valerie A., "Macroeconomic shocks and their propagation," in John B. Taylor and Harald Uhlig, eds., *Handbook of Macroeconomics*, Vol. 2, Elsevier, 2016, chapter 2, pp. 71–162.

Rosa, Carlo, "The high-frequency response of energy prices to U.S. monetary policy: Understanding the empirical evidence," *Energy Economics*, 2014, *45*, 295–303.

Routledge, Bryan R., Duane J. Seppi, and Chester S. Spatt, "Equilibrium Forward Curves for Commodities," *The Journal of Finance*, 2000, *55* (3), 1297–1338.

Samuelson, Paul A., "Stochastic speculative price," *Proceedings of the National Academy of*

Sciences, 1971, *68* (2), 335–337.

Scheinkman, José A. and Jack Schechtman, "A simple competitive model with production and storage," *The Review of Economic Studies*, 1983, *50* (3), 427–441.

Schmitt-Grohé, Stephanie and Martín Uribe, "Finite-State Approximation of VAR Processes: A Simulation Approach," note 2014. Available from `https://www.columbia.edu/~mu2166/tpm/tpm.html`.

Schorfheide, Frank, Dongho Song, and Amir Yaron, "Identifying long-run risks: A Bayesian mixed-frequency approach," *Econometrica*, 2018, *86* (2), 617–654.

Schwartz, Eduardo S., "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging," *The Journal of Finance*, 1997, *52* (3), 923–973.

Scrimgeour, Dean, "Commodity price responses to monetary policy surprises," *American Journal of Agricultural Economics*, 2015, *97* (1), 88–102.

Stachurski, John, *Economic Dynamics: Theory and Computation*, MIT Press, 2009.

______ **and Junnan Zhang**, "Dynamic programming with state-dependent discounting," *Journal of Economic Theory*, 2021, *192*, 105190.

Tauchen, George, "Finite state Markov-chain approximations to univariate and vector autoregressions," *Economics Letters*, 1986, *20* (2), 177–181.

Toda, Alexis Akira, "Perov's contraction principle and dynamic programming with stochastic discounting," *Operations Research Letters*, 2021, *49* (5), 815–819.

Tumen, Semih, Deren Unalmis, Ibrahim Unalmis, and D. Filiz Unsal, "Taxing fossil fuels under speculative storage," *Energy Economics*, 2016, *53*, 64–75.

Wright, Brian D. and Jeffrey C. Williams, "The economic role of commodity storage," *The Economic Journal*, 1982, *92* (367), 596–614.