Economic Dynamics: Solutions to Selected Exercises

This document contains solutions to most of the exercises for the second edition of *Economic Dynamics: Theory and Computation* by John Stachurski.

I have focused on providing the kinds of answers that I thought would be hard to find by searching. (For example, the exercises in Appendix A are all quite standard, since we area treating basic real analysis, and solutions are omitted.)

Solution to Exercise 1.1. The variable X_1 is normally distributed, since X_0 is constant and constant plus normal equals normal. Moreover X_{t+1} is normally distributed whenever X_t is normally distributed because linear combinations of *independent* normal random variables are themselves normal.

Solution to Exercise 2.1. Here is a modification that produces the maximizer:

```
set c = −∞
for x in S do
    if c < f(x) then
        set c = f(x)
        set x* = x
    end
end
print x*
```

The reason the maximizer is more useful is that it provides more information: The maximum is easily evaluated once we have the maximizer but the converse is not true.

Solution to Exercise 2.2. I won't provide a solution to this exercise or the next one, but I encourage you to write the algorithms up in your favorite programming language and test them. It will not be hard to iterate until the program is working correctly.

Solution to Exercise 2.5. Fix $x \in S$ and $z \in (0,1]$. If $\tau(z) = x$, then, since all elements of S are distinct, the definition of τ implies $z \in I(x)$. Conversely, if $z \in I(x)$, then, since all intervals are disjoint, we have $\tau(z) = x$.

Solution to Exercise 3.2. Let $\| \cdot \|$ be a norm on $\mathbb{R}^k$ and fix $x, y \in \mathbb{R}^k$. By the triangle inequality, $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$. Hence $\|x\| - \|y\| \leq \|x - y\|$. Reversing the roles of x and y yields $\|y\| - \|x\| \leq \|x - y\|$. The last two inequalities imply $|\|x\| - \|y\|| \leq \|x - y\|$, as was to be shown.

Solution to Exercise 3.3. Only the triangle inequality is nontrivial to verify. To see that it holds in the case $p = \infty$, fix $x, y \in \mathbb{R}^k$ and $i \leq k$. By the triangle inequality in $\mathbb{R}$ we have $|x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|_\infty + \|y\|_\infty$. Maximizing over i gives the triangle inequality for the norm.

Solution to Exercise 3.4. Let (x_n) and (y_n) be as stated. For any $n \in \mathbb{N}$, the triangle inequality gives $0 \leq \rho(y_n, x) \leq \rho(y_n, x_n) + \rho(x_n, x)$. Since the right hand side converges to zero as $n \to \infty$, we have $\rho(y_n, x) \to 0$, as claimed.

Solution to Exercise 3.43. Suppose that there exists a pair $x, y \in \mathbb{R}$ with $Tx = x$ and $Ty = y$. If $x < y$, then $Tx < Ty$, which contradicts the decreasing property. The case $y < x$ can be ruled out in similar fashion. Hence $x = y$.

Solution to Exercise 3.44. Let $T \colon S \to S$ be nonexpansive. Fix $x \in S$ and $(x_n) \subset S$. We have $0 \leq \rho(Tx_n, Tx) \leq \rho(x_n, x)$, so $x_n \to x$ implies $Tx_n \to Tx$. Hence T is continuous at all $x \in S$.

Solution to Exercise 3.45. Let T be a contraction on S. If $x, y \in S$ are distinct fixed points, then $\rho(x, y) = \rho(Tx, Ty)$ and $\rho(Tx, Ty) < \rho(x, y)$. Contradiction.

Solution to Exercise 3.47. Let S, T be as stated and fix distinct $x, y \in S$. Taking the derivative will convince you that T is increasing on S. Assume without loss of generality that $x < y$. We then have

$$|Tx - Ty| = Ty - Tx = y - x + e^{-y} - e^{-x} < y - x = |x - y|$$

so T is indeed contracting. At the same time, a fixed point of T on S is an $x \in \mathbb{R}_+$ satisfying $x = x + e^{-x}$. Clearly this is impossible.

Solution to Exercise 4.1. Let (S, h), x and x' be as stated. Let $x_t = h^t(x)$, so that $x_t \to x'$. For the sequence $(h(x_t))_{t \geq 1}$, continuity implies that $h(x_t) \to h(x')$. However, $(h(x_t))_{t \geq 1} = (x_t)_{t \geq 2}$, and so $h(x_t) \to x'$ also holds. (Why?) Now we have $h(x_t) \to x'$ and $h(x_t) \to h(x')$. Since limits are unique, it must be that $h(x') = x'$.

Solution to Exercise 4.2. Fix $x \in \operatorname{cl} A$. By the definition of closure, there exists a sequence $(a_n) \subset A$ such that $a_n \to x$. Since $h(A) \subset A$, we have $h(a_n) \in A$ for all n. Therefore, $h(x) = \lim_n h(a_n) \in \operatorname{cl} A$. Hence $h(\operatorname{cl} A) \subset \operatorname{cl} A$, as was to be shown.

Solution to Exercise 4.3. This follows directly from the definition of open sets.

Solution to Exercise 4.4. If x' is another fixed point, then iteration from x' fails to converge to x^*. Contradiction.

Solution to Exercise 4.5. Let (S, h) be as stated and fix $x \in S$. The set $\{h^n(x)\}_{n \in \mathbb{N}}$ is bounded because S is bounded, and therefore every subsequence contains a convergent subsubsequence. Since S is closed, the limit is in S. Therefore $\{h^n(x)\}_{n \in \mathbb{N}}$ is precompact as a subset of S.

Solution to Exercise 4.6. Let (S, h) be as stated and fix $x \in S$. Either $x \leq h(x)$ or $h(x) \leq x$. In the first case, we can apply h to both sides of the inequality to obtain $h(x) \leq h^2(x)$. Continuing in this fashion proves that $(h^n(x))_{n \in \mathbb{N}}$ is increasing. A similar argument shows that, in the case where $h(x) \leq x$, the trajectory is decreasing.

Solution to Exercise 4.7. Here's a counterexample: Take $h(x) = 2x$, in the sense of scalar multiplication. If $x = (-1, 1)$, then $h(x) = (-2, 2)$. Neither $x \leq h(x)$ nor $h(x) \leq x$.

Solution to Exercise 4.8. The relationship $h^t(x) = a^t x + b \sum_{i=0}^{t-1} a^i$ for each t is easily checked by induction. When $|a| < 1$, the first term on the right hand side converges to zero and the second to $x^* := b/(1-a)$. The reader can confirm that $h(x^*) = x^*$.

Solution to Exercise 4.9. The easiest way to prove this is to break it down case by case. For example, if $a = 1$ and $b = 0$, then h is the identity, which has a continuum of fixed points. If $a = 1$ and $b \neq 0$, then a fixed point must satisfy $x = x + b$ for nonzero b, which is impossible. Further details are left to the reader.

Solution to Exercise 4.10. We know that $\mathbb{R}$ is complete and, moreover, $|h(x) - h(y)| = |ax - ay| = |a||x - y|$ for any $x, y \in \mathbb{R}$. As $|a| < 1$, we can apply Banach's fixed point theorem.

Solution to Exercise 4.11. For the first claim, take a Cauchy sequence (x_n) in (S, ρ) and let $y_n = \ln x_n$. You will be able to verify that the Cauchy property of (x_n) in (S, ρ) implies that (y_n) is Cauchy in $(\mathbb{R}, |\cdot|)$. Hence there exists a $y \in \mathbb{R}$ with $|y_n - y| \to 0$. Equivalently, $\rho(x_n, e^y) \to 0$. Hence (x_n) is convergent in (S, ρ) and (S, ρ) is complete. Moreover, $\rho(h(k), h(k')) = \alpha |\ln k - \ln k'| = \alpha \rho(k, k')$ for any $k, k' \in S$, so h is a uniform contraction under the metric ρ. Hence Banach's contraction mapping theorem applies.

Solution to Exercise 4.12. Existence of the maximum follows from Weierstrass' theorem. The bound $\|Ex\| \leq \lambda \|x\|$ is trivial if $x = 0$ so suppose otherwise. Then $\|Ex\| = \|x\| \|Ey\|$ where $y := x / \|x\|$. Since $\|y\| = 1$, we now have $\|Ex\| \leq \|x\| \lambda$, as was to be shown. The global stability result follows from Banach's fixed point theorem when $\lambda < 1$, since $\|Ex - Ey\| = \|E(x-y)\| \leq \lambda \|x - y\|$.

Solution to Exercise 4.13. In view of exercise 4.12, we need only show that $\lambda := \max_{\|x\|=1} \|Ax\| < 1$, where $\| \cdot \| := \| \cdot \|_\infty$. This is true because, when $\|x\| = \max_i |x_i| = 1$,

$$\|Ax\| = \max_i \left| \sum_j a_{ij} x_j \right| \leq \max_i \sum_j |a_{ij}| |x_j| \leq \max_i \sum_j |a_{ij}|.$$

Under the stated condition on row sums, the right hand side is < 1.

Solution to Exercise 4.14. In view of exercise 4.12, we need only show that $\lambda := \max_{\|x\|=1} \|Bx\| < 1$, where $\| \cdot \| := \| \cdot \|_1$. Let $\beta = \max_j \sum_i |b_{ij}|$. When $\|x\| = \sum_j |x_j| = 1$, we have

$$\|Bx\| = \sum_i \left| \sum_j b_{ij} x_j \right| \leq \sum_i \sum_j |b_{ij}| |x_j| \leq \sum_j \sum_i |b_{ij}| |x_j| \leq \beta$$

By assumption, $\beta < 1$, so $\lambda \leq \beta < 1$.

Solution to Exercise 4.15. Let the stated conditions hold and let x^* be the unique fixed point of h in S. Fix $a \in A$. Since (S, h) is globally stable, we have $a_n := h^n(a) \to x^*$ as $n \to \infty$. As $h(A) \subset A$, the sequence (a_n) lies in A. Finally, because A is closed, any limit point of a sequence in A is also in A. Therefore, $x^* \in A$.

Solution to Exercise 4.18. To show that $\hat{g} = \tau \circ g \circ \tau^{-1}$ holds, we can equivalently prove that $\hat{g} \circ \tau = \tau \circ g$. For $x \in \mathbb{R}$, we have $\tau(g(x)) = \ln A + \alpha \ln x$ and $\hat{g}(\tau(x) = \ln A + \alpha \ln x$. Hence $\hat{g} \circ \tau = \tau \circ g$, as was to be shown.

Solution to Exercise 4.19. Let (S, g) and $(\hat{S}, \hat{g})$ be topologically conjugate, with $\hat{g} \circ \tau = \tau \circ g$. The stated equivalence holds because

$$g(x) = x \iff \tau(g(x)) = \tau(x) \iff \hat{g}(\tau(x)) = \tau(x).$$

Solution to Exercise 4.20. From $\hat{g} = \tau \circ g \circ \tau^{-1}$ we have $\hat{g}^2 = \tau \circ g \circ \tau^{-1} \circ \tau \circ g \circ \tau^{-1} = \tau \circ g^2 \circ \tau^{-1}$ and, continuing in the same way (or using induction), $\hat{g}^t = \tau \circ g^t \circ \tau^{-1}$ for all $t \in \mathbb{N}$. Equivalently, $\hat{g}^t \circ \tau = \tau \circ g^t$ for all $t \in \mathbb{N}$. Hence, using continuity of τ and τ^{-1},

$$g^t(x) \to x^* \iff \tau(g^t(x)) \to \tau(x^*) \iff \hat{g}^t(\tau(x)) \to \tau(x^*).$$

Solution to Exercise 4.21. These facts can be established by applying the results of the last two exercises. Details are omitted.

Solution to Exercise 4.23. See the Jupyter code book for solutions to this and other computational exercises.

Solution to Exercise 4.25. Let p be a stochastic kernel on S and let p^t be the t-th order kernel. By definition, p^1 is a stochastic kernel on S. Now suppose the same is true at $t-1$. Then $p^t(x,y) = \sum_{z \in S} p^{t-1}(x,z)p(z,y)$ is nonnegative and, in addition,

$$\sum_{y \in S} p^t(x,y) = \sum_{y \in S} \sum_{z \in S} p^{t-1}(x,z)p(z,y) = \sum_{z \in S} p^{t-1}(x,z) \sum_{y \in S} p(z,y) = \sum_{z \in S} p^{t-1}(x,z).$$

Using the induction hypothesis now completes the proof.

Solution to Exercise 4.26. The defining expression $p^t(x,y) = \sum_{z \in S} p^{t-1}(x,z)p(z,y)$ is just matrix multiplication written out element by element. Regarding these kernels as matrices, we can equivalently write $p^t = p^{t-1}p$. Thus, $p^t(x,y)$ is the (x,y)-th element of the t-th power of p, as was to be shown.

Solution to Exercise 4.27. Fixing stochastic kernel p, as well as $k, j \in \mathbb{N}$ and $x, y \in S$, we have, by lemma 4.2.5,

$$p^{j+k}(x,y) = (\delta_x \mathbf{M}^{j+k})(y) = (\delta_x \mathbf{M}^j \mathbf{M}^k)(y) = \sum_{z \in S} (\delta_x \mathbf{M}^j)(z) p^k(z,y)$$

Since $(\delta_x \mathbf{M}^j)(z) = p^j(x,z)$, we recover the Chapman–Kolmogorov relation.

Solution to Exercise 4.28. This follows easily from the definitions and induction on t. The details are omitted.

Solution to Exercise 4.29. See the code book for solutions to this and other computational exercises.

Solution to Exercise 4.35. At one billion paths per second, total run time is $10^{100}/10^9 = 10^{91}$ seconds. There are around 3×10^7 seconds in year, so run time in years is more than 10^{83}. The universe is estimated to be around 4×10^{10} years old.

Solution to Exercise 4.37. Fix $\psi \in \mathscr{P}(S)$. At each $y \in S$, we have $\psi\mathbf{M}(y) = \sum_{x \in S} p(x,y)\psi(x)$. Since p is a stochastic kernel, easy arguments confirm that $\psi\mathbf{M}(y) \geq 0$ and $\sum_{y \in S} \psi\mathbf{M}(y) = 1$. Hence $\psi\mathbf{M} \in \mathscr{P}(S)$.

Solution to Exercise 4.38. Let ψ_i and Ψ_i be as defined in the exercise, $i = 1, 2$. Let $D = \{x \in S : \psi_1(x) \geq \psi_2(x)\}$. For any $A \subset S$, we can decompose the sum over

$A = (A \cap D) \cup (A \cap D^c)$ and apply the triangle inequality to get

$$
\begin{aligned}
|\Psi_1(A) - \Psi_2(A)| &\leq \sum_{x \in A \cap D} |\psi_1(x) - \psi_2(x)| + \sum_{x \in A \cap D^c} |\psi_1(x) - \psi_2(x)| \\
&= \sum_{A \cap D} (\psi_1(x) - \psi_2(x)) + \sum_{A \cap D^c} (\psi_2(x) - \psi_1(x)) \\
&\leq \sum_{D} (\psi_1(x) - \psi_2(x)) + \sum_{D^c} (\psi_2(x) - \psi_1(x))
\end{aligned}
$$

The right hand side evaluates to $\Psi_1(D) - \Psi_2(D) = |\Psi_1(D) - \Psi_2(D)|$. As a consequence of this calculation, we see that

$$\sup_{A \subset S} |\Psi_1(A) - \Psi_2(A)| = |\Psi_1(D) - \Psi_2(D)|$$

Now observe that

$$\|\psi_1 - \psi_2\| = \sum_{D} (\psi_1(x) - \psi_2(x)) + \sum_{D^c} (\psi_2(x) - \psi_1(x))$$

and, moreover, since $\sum_{x \in S} (\psi_1(x) - \psi_2(x)) = 0$,

$$0 = \sum_{D} (\psi_1(x) - \psi_2(x)) + \sum_{D^c} (\psi_1(x) - \psi_2(x)) = \sum_{D} (\psi_1(x) - \psi_2(x)) - \sum_{D^c} (\psi_2(x) - \psi_1(x))$$

Combining these results gives $\|\psi_1 - \psi_2\| = 2 \sum_D (\psi_1(x) - \psi_2(x)) = 2s(\psi_1, \psi_2)$.

Solution to Exercise 4.39. Fix $\psi, \psi' \in \mathscr{P}(S)$ and Markov operator $\mathbf{M}$ corresponding to stochastic kernel p. We have

$$d_1(\psi \mathbf{M}, \psi' \mathbf{M}) = \sum_{y} \left| \sum_{x} p(x,y) \psi(x) - \sum_{x} p(x,y) \psi'(x) \right| \leq \sum_{y} \sum_{x} p(x,y) |\psi(x) - \psi'(x)|$$

Reversing the other of the sums and using $\sum_y p(x,y) = 1$ gives the desired conclusion.

Solution to Exercise 4.40. If $p = \mathbf{I}_N$, the $N \times N$ identity, then every distribution is stationary.

Solution to Exercise 4.41. If ψ is a stationary distribution, then $\psi(\mathbf{I}_N - p + \mathbb{1}_{N \times N}) = \psi \mathbb{1}_{N \times N} = \mathbb{1}_N$. The restriction that the elements of ψ sum to 1 is imposed by the last equality.

Solution to Exercise 4.44. Let $\psi^* = (a, b)$. If ψ^* is stationary, then, by $\psi^* \mathbf{M} = \psi^*$ and the choice of p, we must have $(a, b) = (b, a)$. Hence $a + b = 1$ and $a = b$. This yields $a = b = 1/2$. For a counterexample to the global stability statement, try iterating on $\psi = (1, 0)$.

Solution to Exercise 4.45. This follows directly from the definition and $\sum_x p(x,y) = 1$ for all x.

Solution to Exercise 4.46. It is clear from the definition that $p(x,dy) = q \in \mathscr{P}(S)$ for all $x \in S$ implies $\alpha(p) = 1$. Regarding the converse, suppose to the contrary that $p(x,dy)$ and $p(x',dy)$ are distinct for some $x, x' \in S \times S$. Since both $p(x,dy)$ and $p(x',dy)$ are distributions, we can select a $z \in S$ with $p(x,z) < p(x',z)$. Hence

$$\alpha(p) \leq \sum_{y \in S} p(x,y) \wedge p(x',y) \leq \sum_{y \neq z} p(x',y) + p(x,z) < \sum_{y \in S} p(x',y) = 1.$$

Solution to Exercise 4.48. Evidently

$$\alpha(p) > 0 \iff \forall\,(x,x') \in S \times S,\ \exists\, y \in S \text{ s.t. } p(x,y) \wedge p(x',y) > 0$$

The statement on the right means precisely that $p(x,dy)$ and $p(x',dy)$ overlap.

Solution to Exercise 4.49. This follows immediately from exercise 4.47, since p^t is the periodic kernel when t is odd and the identity when t is even.

Solution to Exercise 4.50. Suppose $\min_{x \in S} p^t(x,\bar{y}) =: \epsilon > 0$ for some $\bar{y} \in S$. Under this condition, a simple calculation yields $\alpha(p^t) \geq \epsilon$. Hence, by theorem 4.3.5, global stability holds.

Solution to Exercise 4.51. Shifting a minimum inside a sum makes the value (weakly) smaller, since we can minimize term by term. Because of this,

$$\alpha(p^t) = \min_{(x,x')} \sum_{y \in S} p^t(x,y) \wedge p^t(x',y) \geq \sum_{y \in S} \min_{(x,x')} p^t(x,y) \wedge p^t(x',y) = \sum_{y \in S} \min_x p^t(x,y).$$

Hence, if the condition of Stokey and Lucas holds, then $\alpha(p^t) > 0$ and $(\mathscr{P}(S), \mathbf{M})$ is globally stable.

Solution to Exercise 4.52. To show that part 2 implies part 1, suppose $\alpha(p^t) > 0$ for some $t \in \mathbb{N}$. By theorem 4.3.4 and Banach's contraction mapping theorem, $(\mathscr{P}(S), \mathbf{M}^t)$ is globally stable. (We are also using lemma 4.2.5 from page 82 to connect p^t and $\mathbf{M}^t$.) But then $(\mathscr{P}(S), \mathbf{M})$ is globally stable, by lemma 4.1.5 on page 67.

To show that part 1 implies part 2, let ψ^* be the stationary distribution. Note that $\exists \bar{y} \in S$ with $\psi^*(\bar{y}) > 0$. By global stability, $p^t(x,\bar{y}) \to \psi^*(\bar{y})$ for any x. Using finiteness of S, we can obtain a $t \in \mathbb{N}$ with $\min_{x \in S} p^t(x,\bar{y}) > 0$. But then $\alpha(p^t) > 0$, by exercise 4.50.

Solution to Exercise 4.53. In view of exercise 4.48, it suffices to provide a pair of rows of p_Q that fail to overlap (when regarded as distributions). This is true for the first and last rows of the matrix.

Solution to Exercise 4.54. Applying exercise 4.48, we have $\alpha(p_q^{23}) > 0$ because any two rows of p_q^{23} overlap.

Solution to Exercise 4.58. Let p be the corresponding stochastic kernel. In view of exercise 4.48, it suffices to show that any two rows of p overlap.

Notice that inventory shifts to zero in one step whenever demand is greater than Q. Given our definition of b, this is a positive probability event. Hence $p(x, 0) > 0$ for all $x \in S$. As a result, any two rows overlap.

Solution to Exercise 4.62. Let p be the identity on S and let x, y be distinct points in S. Both δ_x and δ_y are stationary for p. But (X_t) started at x never visits y. Hence δ_y does not match the fraction of time the chain spends in each state.

Solution to Exercise 5.1. Fix $\sigma \in \Sigma$ and $(x, y) \in S \times S$. Letting Z be a draw from ϕ, The kernel corresponding to the SRS (5.1) obeys

$$p_\sigma(x, y) = \mathbb{P}\{\sigma(x) + Z = y\} = \mathbb{P}\{Z = y - \sigma(x)\} = \phi(y - \sigma(x))$$

Solution to Exercise 5.5. Some thought will convince you that $p(x, y) > 0$ for every $(x, y) \in S \times S$. For example, if $y \geq B(x)$, then the state travels from x to y whenever $W^v_{t+1} = 0$ and $W^u_{t+1} = y - B(x)$. This is a positive probability event. It follows directly from strict positivity of p that $\alpha(p) > 0$. Hence global stability holds.

Solution to Exercise 6.1. These results follow easily from the restrictions on the production function and the fact that $Z := (0, \infty)$, so every shock is positive.

Solution to Exercise 6.2. Code is in the Jupyter code book. Since the draws $\{k^i_t\}_{i=1}^n$ are IID across i, the sample mean converges to the mean, as per the LLN result in theorem 4.3.6.

Solution to Exercise 6.10. Let f_n be the kernel density estimate in (6.10). Clearly f_n is nonnegative. Also, since K is a density, for any $y \in \mathbb{R}$ and $\delta > 0$, applying the change of variable $z = (x - y)/\delta$ yields

$$\int K\left(\frac{x - y}{\delta}\right) dx = \int K(z)\delta dz = \delta$$

It now follows from the definition of f_n that $\int f_n(x) dx = 1$ for all n.

Solution to Exercise 6.17. The proof is straightforward: If u and v are arbitrary elements of the metric space (U, d) and M and N have the stated properties, then

$$d(MNu, MNv) \leq d(Nu, Nv) \leq \rho d(u, v)$$

This is all we need to show.

Solution to Exercise 6.20. Since continuity is directly assumed, we only need to check that functions in $\mathscr{C}$ are bounded. But this is obvious because $S = [a, \infty)$ and each $p \in \mathscr{C}$ is decreasing. Hence $p(x) \leq p(a) < \infty$ for all $x \in S$.

Solution to Exercise 6.21. These claims follow from the fact that convergence in d_∞ preserves weak inequalities. For example, suppose $h_n \in \mathscr{C}$ for all n and $d_\infty(h_n, h) \to 0$ for some function $h \in bcS$. Fixing $x \in S$ and noting that uniform convergence implies pointwise convergence, we have $h_n(x) \geq P(x)$ for all n and $h_n(x) \to h(x)$. Hence $h(x) \geq P(x)$. Since x is arbitrary, $h \geq P$ on S.

Solution to Exercise 6.22. This is just a matter of checking the definition. Details are omitted.

Solution to Exercise 6.23. Let h_1 and h_2 be as stated, with fixed points x_1 and x_2. Suppose to the contrary that $x_1 > x_2$. Then, since h_1 is decreasing, $h_1(x_1) \leq h_1(x_2)$. Because $h_1 \leq h_2$ and x_i is a fixed point of h_i, this yields $x_1 \leq h_2(x_2) = x_2$. Contradiction.

Solution to Exercise 6.24. This is immediate because $v(x)$ is the maximum of $\alpha \int p(z)\phi(z)dz$ and $P(x)$. Hence if $\alpha \int p(z)\phi(z)dz \leq P(x)$, then $v(x) = P(x)$. Given that $r \in [P(x), v(x)]$, we now have $r = P(x)$.

Solution to Exercise 6.25. We are considering the unique $r \in [P(x), v(x)]$ such that (6.31) holds. To prove that

$$r = \alpha \int p(\alpha(x - D(r)) + z)\phi(z)dz$$

as required by the exercise, it suffices to show that $r > P(x)$, for then the claim will be true by (6.31). But $r > P(x)$ must hold. To see this, suppose to the contrary that $r = P(x)$. By (6.31), this leads to

$$r = \max\left\{\alpha \int p(z)\phi(z)dz, P(x)\right\}$$

At the same time, our hypothsis is $\alpha \int p(z)\phi(z)dz > P(x)$, whence $r > P(x)$. Contradiction.

Solution to Exercise 6.27. When we compare P and p^*, we understand that the former is the value of a commodity without storage, while the latter is the value of the same commodity when we add the possibility of storage. The commodity is more valuable when it can be stored. (The degree of storability is parameterized by α, so higher α pushes up p^*.)

Solution to Exercise 7.1. The claim is that if $A \subset B$, then $\lambda(A) \leq \lambda(B)$. As in the main text, let C_F be the set of coverings of F. In addition, let H_F be the set $\{\sum_n \ell(I_n) : (I_n) \in C_F\}$. By $A \subset B$, every covering of B is also a covering of A. Hence $C_B \subset C_A$, and, in turn, $H_B \subset H_A$. By lemma A.2.16 on page 335, $H_B \subset H_A$ implies $\inf H_A \leq \inf H_B$. That is, $\lambda(A) \leq \lambda(B)$.

Solution to Exercise 7.2. The claim is that if A and B are any two subsets of $\mathbb{R}^k$, then $\lambda(A \cup B) \leq \lambda(A) + \lambda(B)$. To see this, fix $\epsilon > 0$ and choose covers $(I_n^A)_{n\geq 1}$ and $(I_n^B)_{n\geq 1}$ of A and B respectively such that $\sum_n \ell(I_n^A) \leq \lambda(A) + \epsilon/2$ and $\sum_n \ell(I_n^B) \leq \lambda(B) + \epsilon/2$. Clearly $(\cup_n I_n^A) \cup (\cup_n I_n^B)$ contains $A \cup B$, so $(I_n^A, I_n^B)_{n\geq 1}$ is a cover of $A \cup B$.[4] By the definition of λ, we then have

$$\lambda(A \cup B) \leq \sum_n \ell(I_n^A) + \sum_n \ell(I_n^B) \leq \lambda(A) + \lambda(B) + \epsilon$$

Since ϵ was arbitrary, the claim has been established.

Solution to Exercise 7.3. The claim is that for any $(A_n) \subset \mathfrak{P}(\mathbb{R})$ we have $\lambda(\cup_n A_n) \leq \sum_n \lambda(A_n)$. To see this, fix any such (A_n), and any $\epsilon > 0$. Associate to each A_n a cover $(I_j^n)_{j\geq 1}$ such that $\sum_j \ell(I_j^n) \leq \lambda(A_n) + \epsilon 2^{-n}$. The family $(I_j^n)_{n,j\geq 1}$ is countable (see the figure in the proof of theorem A.1.3 on page 324) and covers $\cup_n A_n$. The rest of the proof is similar to that of exercise 7.2.

Solution to Exercise 7.4. In view of (7.3), to show that $\mathbb{R}^k \in \mathscr{L}$, we need to demonstrate that $\lambda(B) = \lambda(B \cap \mathbb{R}^k) + \lambda(B \cap (\mathbb{R}^k)^c)$ for arbitrary $B \subset \mathbb{R}$. Since $(\mathbb{R}^k)^c = \emptyset$, this equality will hold provided that $\lambda(\emptyset) = 0$. This is indeed the case, since $\mathscr{J}$ was allowed to contain empty intervals in its definition, and we set $\ell(\emptyset) = 0$.

The proof that $\emptyset \in \mathscr{L}$ is similar and hence omitted. Thus it remains only to show that if $N \subset \mathbb{R}$ and $\lambda(N) = 0$, then $N \in \mathscr{L}$. To this end, pick any such N and any $B \subset \mathbb{R}$. The claim will be established if we can show that

$$\lambda(B) \geq \lambda(B \cap N) + \lambda(B \cap N^c)$$

[4] If you want to be more formal and insist that a cover is a single sequence $(J_n)_{n\geq 1}$, then you can construct such a sequence by letting the odd elements $J_1, J_3, J_5, \ldots$ equal $(I_n^A)_{n\geq 1}$ and the even elements $J_2, J_4, J_6, \ldots$ equal $(I_n^B)_{n\geq 1}$.

(The reverse inequality holds by subadditivity.) By monotonicity and $\lambda(N) = 0$, we have $\lambda(B \cap N) = 0$, so the claim reduces to $\lambda(B) \geq \lambda(B \cap N^c)$. Since $B \cap N^c \subset B$, another application of monotonicity yields the desired result.

Solution to Exercise 7.5. Suppose that countable additivity holds. The claim is that $\lambda(\cup_{n=1}^N A_n) = \sum_{n=1}^N \lambda(A_n)$ for any finite collection of disjoint sets $(A_n)_{n=1}^N$. Let $(A_n)_{n=1}^N$ be such a colleciton. The desired equality can be obtained by applying countable additivity to the sequence $(B_n)_{n \geq 1}$, where $B_n := A_n$ for $n \leq N$ and $B_n := \emptyset$ for $n > N$.

Solution to Exercise 7.6. Let A and B be two sets in $\mathscr{L}$ with $A \subset B$ and $\lambda(B) < \infty$. The claim is that $\lambda(B \setminus A) = \lambda(B) - \lambda(A)$. To see this, observe that $B \setminus A$ and A are disjoint sets with union B. Hence, by additivity, $\lambda(B \setminus A) + \lambda(A) = \lambda(B)$. Since all terms are finite, we can rearrange to obtain the desired equality.

Solution to Exercise 7.7. The claim is that $\lambda(\mathbb{R}^k) = \infty$. Since $\lambda(\mathbb{R}^k)$ is a well-defined element of $[0, \infty]$, it suffices to show that $\lambda(\mathbb{R}^k)$ is bigger than any real number. To this end, consider the intervals $I_n := (0, n]^k := (0, n] \times \cdots \times (0, n]$. By monotonicity (exercise 7.1) we have $\lambda(\mathbb{R}) \geq \lambda(I_n)$ for all n. By lemma 7.1.1 we have $\lambda(I_n) = \ell(I_n) = n^k$. Hence $\lambda(\mathbb{R}) \geq n^k$ for all $n \in \mathbb{N}$, competing the proof.

Solution to Exercise 7.8. The claim is that countable sets have zero measure. To see this, A be any countable set, and let $(a_n)_{n \geq 1}$ be an enumeration of A consisting only of distinct points. By countable additivity and the fact that singletons have zero measure, we have $\lambda(A) = \sum_n \lambda(\{a_n\}) = 0$.

Solution to Exercise 7.9. Let $\mathscr{S}$ be a σ-algebra on S. The claim is that both $S \in \mathscr{S}$ and $\emptyset \in \mathscr{S}$. Since $\mathscr{S}$ is closed under complements, it is enough to check that $S \in \mathscr{S}$. Since $\mathscr{S}$ is nonempty by definition, there exists at least one $A \in \mathscr{S}$. By the definition of $\mathscr{S}$, we then have $A^c \in \mathscr{S}$, and therefore $A \cup A^c \in \mathscr{S}$. But $A \cup A^c = S$.

Solution to Exercise 7.10. The claim is that if $\{\mathscr{S}_\alpha\}_{\alpha \in \Lambda}$ is any collection of σ-algebras on S, then their intersection $\mathscr{S} := \cap_\alpha \mathscr{S}_\alpha$ is itself a σ-algebra on S. Let's just check that $\mathscr{S}$ is closed under countable unions. To see that this is so, let (A_n) be a sequence of sets with $A_n \in \mathscr{S}$ for all n. The statement $A_n \in \mathscr{S}$ is equivalent to $A_n \in \mathscr{S}_\alpha$ for all α. Fixing any such α, we can use the σ-algebra property of $\mathscr{S}_\alpha$ to obtain $\cup_n A_n \in \mathscr{S}_\alpha$. Since α was arbitrary, we then have $\cup_n A_n \in \mathscr{S}$.

Solution to Exercise 7.11. The first claim is that if $\mathscr{C}$ is a σ-algebra, then $\sigma(\mathscr{C}) = \mathscr{C}$. To see this, let $\mathscr{C}$ be any σ-algebra. On one hand, we have $\sigma(\mathscr{C}) \subset \mathscr{C}$, because $\mathscr{C}$ is a σ-algebra containing $\mathscr{C}$, and, by definition, $\sigma(\mathscr{C})$ is contained in every such collection. On the other hand, $\mathscr{C} \subset \sigma(\mathscr{C})$ also holds, because, by definition, $\sigma(\mathscr{C})$ is a σ-algebra *containing* $\mathscr{C}$.

Next, let $\mathscr{C}$ and $\mathscr{D}$ be two collections of sets with $\mathscr{C} \subset \mathscr{D}$. The claim is that $\sigma(\mathscr{C}) \subset \sigma(\mathscr{D})$. To see this, just observe that $\sigma(\mathscr{D})$ is, by definition, a σ-algebra containing $\mathscr{D}$, which in turn contains $\mathscr{C}$. But $\sigma(\mathscr{C})$ is the the smallest σ-algebra containing $\mathscr{C}$. Hence $\sigma(\mathscr{C}) \subset \sigma(\mathscr{D})$.

Solution to Exercise 7.12. To see that $\mathscr{B}(S)$ contains the closed subsets of the metric space S, let F be any closed subset of S. Since $G = F^c$ is open, $G \in \mathscr{B}(S)$. Since $\mathscr{B}(S)$ is a σ-algebra, and therefore closed under complementation, it follows that $F = G^c$ is again in $\mathscr{B}(S)$.

To see that $\mathbb{Q} \in \mathscr{B}(\mathbb{R})$, observe that any singleton is closed, and hence, for a rational number $r \in \mathbb{Q}$, we have $\{r\} \in \mathscr{B}(S)$. Since $\mathbb{Q}$ can be expressed as the countable union of such sets, and since $\mathscr{B}(S)$ is closed under countable unions, we conclude that $\mathbb{Q} \in \mathscr{B}(S)$.

Solution to Exercise 7.13. Let $\mathscr{A}$ be the set of all open intervals $(a, b) \subset \mathbb{R}$. The claim is that $\sigma(\mathscr{A}) = \mathscr{B}(\mathbb{R})$. To see this, observe first that since $\mathscr{A} \subset \mathscr{O}$, we must have $\sigma(\mathscr{A}) \subset \sigma(\mathscr{O}) = \mathscr{B}(\mathbb{R})$. To show that $\sigma(\mathscr{O}) \subset \sigma(\mathscr{A})$ it is sufficient to prove that $\sigma(\mathscr{A})$ contains the open sets. (Recall that $\sigma(\mathscr{O})$ is, by definition, contained in every σ-algebra that contains the open sets.) As mentioned in the hint to the exercise, every open subset of $\mathbb{R}$ can be expressed as a countable union of open intervals. Since $\sigma(\mathscr{A})$ contains all the open intervals and is closed under countable unions, we conclude that $\sigma(\mathscr{A})$ contains the open sets.

Solution to Exercise 7.14. Let μ be a function from $\mathscr{S}$ to $[0, \infty]$ such that μ is countably additive on $\mathscr{S}$ and $\mu(A) < \infty$ for some $A \in \mathscr{S}$. The claim is that $\mu(\emptyset) = 0$. To see this, just observe that since A is the disjoint union of $\emptyset$ and A, we have $\mu(A) = \mu(\emptyset) + \mu(A)$. Since $\mu(A)$ is finite, we can cancel to obtain $\mu(\emptyset) = 0$.

Solution to Exercise 7.15. The claim is that if μ is a measure on $(S, \mathscr{S})$, $E, F \in \mathscr{S}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$. To see this, suppose first that $\mu(F) = \infty$. In this case we have nothing to prove. So suppose instead that $\mu(F)$ is finite. Applying $F = E \cup (F \setminus E)$, we have $\lambda(F) = \lambda(E) + \lambda(F \setminus E)$. All terms are nonnegative, and the desired inequality follows.

Solution to Exercise 7.16. Let μ be a measure on $(S, \mathscr{S})$, and let $A, B \in \mathscr{S}$. The claim is that $\mu(A \cup B) \leq \mu(A) + \mu(B)$. To see this, note that $A \cup B$ can also be written as the disjoint union $(A \setminus B) \cup B$. By additivity and monotonicity (exercise 7.15), we have

$$\lambda(A \cup B) = \lambda(A \setminus B) + \lambda(B) \leq \lambda(A) + \lambda(B)$$

Solution to Exercise 7.17. Let $(A_n)_{n \geq 1}$ be a sequence in $\mathscr{S}$, and let μ be a measure on $\mathscr{S}$. The first claim is that if $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$. To see this, let $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. The sequence (B_n) is disjoint with $\cup_{n=1}^k B_n = A_k$ and $\cup_n B_n = A$. Applying countable additivity to this sequence, we have

$$\mu(A) = \mu(\cup_n B_n) = \lim_{k \to \infty} \sum_{n=1}^{k} \mu(B_n) = \lim_{k \to \infty} \mu(A_k)$$

as was to be shown.

The second claim is that if $\mu(A_1) < \infty$ and $A_n \downarrow A$, then $\mu(A_n) \downarrow \mu(A)$. To see this, consider the sequence (B_n) defined by $B_n = A_1 \setminus A_n$. It is not difficult to check that the sequence (B_n) is increasing, with $\cup_n B_n = A_1 \setminus A$. Hence, by the preceding result, $\mu(B_n) \uparrow \mu(A_1 \setminus A)$. Given that $\mu(A_1) < \infty$, we can apply exercise 7.6 to obtain $\mu(A_1) - \mu(A_n) \uparrow \mu(A_1) - \mu(A)$, or, equivalently, $\mu(A_n) \downarrow \mu(A)$.

Solution to Exercise 7.18. The claim is that the set function $\mu(A) = \sum_{j \in A} a_j$ is a measure on $(\mathbb{N}, \mathfrak{P}(\mathbb{N}))$. The condition $\mu(\emptyset) = 0$ is obvious. Regarding countable additivity, let (A_n) be a disjoint sequence of subsets of $\mathbb{N}$. As usual, let $\mathbb{1}\{P\}$ be the indicator function, which is one if statement P is true and zero if it's false. Note that $\mu(A) = \sum_{j \geq 1} \mathbb{1}\{j \in A\} a_j$. Using disjointness, we have $\mathbb{1}\{j \in \cup_n A_n\} = \sum_n \mathbb{1}\{j \in A_n\}$ for any j. (Convince yourself that the right-hand size is zero when the left-hand size is zero, and one when it is one.) As a result,

$$\mu(\cup_n A_n) = \sum_j \mathbb{1}\{j \in \cup_n A_n\} a_j$$

$$= \sum_j \sum_n \mathbb{1}\{j \in A_n\} a_j = \sum_n \sum_j \mathbb{1}\{j \in A_n\} a_j = \sum_n \mu(A_n)$$

Here the third equality holds because $\sum_n \sum_m b_{n,m} = \sum_m \sum_n b_{n,m}$ whenever the summands $b_{n,m}$ are nonnegative.

Solution to Exercise 7.19. The claim is that $\delta_x(A) := \mathbb{1}_A(x) =: \mathbb{1}\{x \in A\}$ is a probability measure on $(S, \mathscr{S})$. That $\delta_x(S) = 1$ is obvious. The claim $\delta_x(\emptyset) = 0$ does not need to be checked (exercise 7.14). Regarding countable additivity, let (A_n) be a disjoint sequence in $\mathscr{S}$. We saw in the solution to exercise 7.18 that $\mathbb{1}\{x \in \cup_n A_n\} = \sum_n \mathbb{1}\{x \in A_n\}$. In other words, $\delta_x(\cup_n A_n) = \sum_n \delta_x(A_n)$, as was to be shown.

Solution to Exercise 7.20. The claim is that $F(x) = \mu((-\infty, x])$ is a cumulative distribution function on $\mathbb{R}$. Nonnegativity of F is obvious. To see that right-continuity holds, let (x_n) be a real sequence with $x_n \downarrow x$. Let $A_n := (-\infty, x_n]$. It is not difficult to check that $A_n \downarrow A := (-\infty, x]$. Hence, by exercise 7.17, we have $\mu(A_n) \downarrow \mu(A)$, or $F(x_n) \downarrow F(x)$. Since x was arbitrary, F is right-continuous on $\mathbb{R}$. The proofs that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ are similar and left to you, the reader.

Solution to Exercise 7.21. The claim is that $\lambda(\mathbb{1}_{\mathbb{Q}}) = 0$. This follows directly from the fact that $\mathbb{Q}$ has zero Lebesgue measure (exercise 7.8) and the definition of the integral for simple functions on page 170. In particular, $\lambda(\mathbb{1}_{\mathbb{Q}}) = \lambda(\mathbb{Q}) = 0$.

Solution to Exercise 7.22. We have $s, s' \in s\mathscr{S}^+$ and $\gamma \geq 0$. The first claim is that $\gamma s \in s\mathscr{S}^+$ and $\mu(\gamma s) = \gamma\mu(s)$. This is straightforward, since for $s = \sum_{n=1}^N \alpha_n \mathbb{1}_{A_n}$ we have

$$\gamma s(x) = \gamma \sum_{n=1}^N \alpha_n \mathbb{1}_{A_n}(x) = \sum_{n=1}^N \gamma\alpha_n \mathbb{1}_{A_n}(x)$$

(In what follows, the argument x is usually omitted.) It is now clear that $\gamma s \in s\mathscr{S}^+$, and

$$\mu(\gamma s) = \sum_{n=1}^N \gamma\alpha_n \mu(A_n) = \gamma \sum_{n=1}^N \alpha_n \mu(A_n) = \gamma\mu(s)$$

The second claim is that $s + s' \in s\mathscr{S}^+$ and $\mu(s+s') = \mu(s) + \mu(s')$. We prove it only for $s = \alpha\mathbb{1}_A$ and $s' = \beta\mathbb{1}_B$, where $A, B \in \mathscr{S}$. A little thought will convince you that

$$s + s' = \alpha\mathbb{1}_{A\setminus B} + (\alpha+\beta)\mathbb{1}_{B\cap A} + \beta\mathbb{1}_{B\setminus A} \tag{7.12}$$

These three sets are disjoint, and the constants are all nonnegative, so $s + s' \in s\mathscr{S}^+$ as claimed. Moreover, by (7.12) and additivity of μ,

$$\begin{aligned}
\mu(s+s') &= \alpha\mu(A\setminus B) + (\alpha+\beta)\mu(B\cap A) + \beta\mu(B\setminus A) \\
&= \alpha\{\mu(A\setminus B) + \mu(B\cap A)\} + \beta\{\mu(B\setminus A) + \mu(B\cap A)\} \\
&= \alpha\mu\left((A\setminus B)\cup(B\cap A)\right) + \beta\mu\left((B\setminus A)\cup(B\cap A)\right) \\
&= \alpha\mu(A) + \beta\mu(B)
\end{aligned}$$

The last expression is just $\mu(s) + \mu(s')$, and the proof is done.

The last claim is monotonicity: $s \leq s'$ implies $\mu(s) \leq \mu(s')$. We prove it only for $s = \alpha\mathbb{1}_A$ and $s' = \beta\mathbb{1}_B$, where $A, B \in \mathscr{S}$. The general case can be found in any text on measure theory. To this end, let s and s' be as above. Note that $\alpha, \beta \geq 0$ by assumption. If $\beta = 0$, then $s' = 0$ and hence $\alpha = 0$, in which $\mu(s) = \mu(s') = 0$. If, on the other hand, $\beta > 0$, then we must have both $\alpha \leq \beta$ and $A \subset B$, as any other possibility would contradict $s \leq s'$. Hence $\mu(A) \leq \mu(B)$, and $\mu(s) = \alpha\mu(A) \leq \beta\mu(B) = \mu(s')$.

Solution to Exercise 7.24. The first claim is that every $f\colon S \to \mathbb{R}$ is $\mathfrak{P}(S)$-measurable. To see this, we only need to check that $f^{-1}(B) \in \mathfrak{P}(S)$ for arbitrary $B \in \mathscr{B}(\mathbb{R})$. This is trivial, because $f^{-1}(B)$ is a subset of S by definition. The second claim is that for $\mathscr{S} :=$

$\{S, \emptyset\}$, only the constant functions are $\mathscr{S}$-measurable. To see this, let $f(x) = \alpha \in \mathbb{R}$ for all $x \in S$. Pick any $B \in \mathscr{B}$. Suppose first that $\alpha \in B$. In this case, $f^{-1}(B) = S$, and $S \in \mathscr{S}$. On the other hand, if $\alpha \notin B$, then $f^{-1}(B) = \emptyset$, which is once again an element of $\mathscr{S}$. Finally, to see that any nonconstant f is not $\mathscr{S}$-measurable, let f take at least two distinct values α and β. Let $B \in \mathscr{B}(\mathbb{R})$ contain α but not β. Then $f^{-1}(B)$ is neither the empty set nor the whole set S. Hence $f^{-1}(B) \notin \mathscr{S}$, and f is not $\mathscr{S}$-measurable.

Solution to Exercise 7.25. The claim is that for arbitrary measurable space $(S, \mathscr{S})$, we have $s\mathscr{S} \subset m\mathscr{S}$. To see this, let s be any element of $s\mathscr{S}$. Recall from the definition that $s = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{A_n}$, where the sets $A_1, \ldots, A_N$ are nonempty, disjoint and $A_n \in \mathscr{S}$ for all n. Pick any $B \in \mathscr{B}(\mathbb{R})$. Let I be all n in $1, \ldots, N$ such that $\alpha_n \in B$. Then $f^{-1}(B) = \cup_{n \in I} A_n$. Since $A_n \in \mathscr{S}$ for all n and $\mathscr{S}$ is a σ-algebra, we conclude that $f^{-1}(B) \in \mathscr{S}$, and hence $s \in m\mathscr{S}$.

Solution to Exercise 7.26. Let S be a metric space, and let $f \colon S \to \mathbb{R}$ be continuous. The claim is that f is Borel measurable, in the sense that elements of $\mathscr{B}(\mathbb{R})$ are pulled back into elements of $\mathscr{B}(S)$. To see this, let $\mathscr{O}$ be the open sets of $\mathbb{R}$. By definition, $\mathscr{O}$ is a generating class of $\mathscr{B}(\mathbb{R})$, and hence, by lemma 7.2.3 on page 173, it is enough to show that $f^{-1}(O) \in \mathscr{B}(S)$ for all $O \in \mathscr{O}$. By theorem 3.1.10 on page 48, we know that $f^{-1}(O)$ is an open subset of S. But $\mathscr{B}(S)$ contains all the open sets, so we are done.

Solution to Exercise 7.27. The claim is that if $f \colon \mathbb{R} \to \mathbb{R}$ is either increasing or decreasing, then f is Borel measurable. Let's check the increasing case, since the decreasing case is very similar. To this end, recall that f will be Borel measurable if $\{f \leq b\} \in \mathscr{B}(\mathbb{R})$ for all $b \in \mathbb{R}$. Fix any $b \in \mathbb{R}$, and consider the set $\{f \leq b\} = \{x \in \mathbb{R} : f(x) \leq b\}$. A little thought will convince you that this set is either of the form $(-\infty, a)$ or $(-\infty, a]$. The first set is open, and hence Borel measurable. The second set is closed, and closed sets are also Borel measurable (theorem 7.1.7 on page 163). Hence f is Borel measurable as claimed.

Solution to Exercise 7.28. The claim is that if $(S, \mathscr{S})$ is a measurable space, if $(f_n) \subset m\mathscr{S}$, and if $f = \sup_n f_n$ is finite (i.e., real-valued at each $x \in S$), then $f \in m\mathscr{S}$. To see this, fix any $b \in \mathbb{R}$. From the definition of the supremum we have

$$\{f \leq b\} = \{x \in S : f(x) \leq b\} = \cap_n \{x \in S : f_n(x) \leq b\} \in \mathscr{S}$$

The result now follows from lemma 7.2.4.

Solution to Exercise 7.29. Let $f \in m\mathscr{S}$. The claim is that $|f| \in m\mathscr{S}$. To see this, fix $b \in \mathbb{R}$. By lemma 7.2.4 on page 173, it is enough to show that $\{|f| \leq b\} \in \mathscr{S}$. Clearly $\{|f| \leq b\} = \{f \leq b\} \cap \{f \geq -b\}$. The intersection is in $\mathscr{S}$ by the measurability of f and the fact that $\mathscr{S}$ is a σ-algebra.

Solution to Exercise 7.30. Let $\gamma \in \mathbb{R}_+$ and $f \in m\mathscr{S}^+$. The first claim is that $\mu(\gamma f) = \gamma\mu(f)$. To see this, let $(s_n) \subset s\mathscr{S}^+$ with $s_n \uparrow f$. Clearly $\gamma s_n \uparrow \gamma f$ also holds. Recalling the definition of the integral in (7.13) and proposition 7.2.1 on page 171, we have

$$\mu(\gamma f) = \lim_{n\to\infty} \mu(\gamma s_n) = \lim_{n\to\infty} \gamma\mu(s_n) = \gamma \lim_{n\to\infty} \mu(s_n) = \gamma\mu(f)$$

A subtle point here is that, as discussed after the definition of the integral was given, if (s'_n) is *any* sequence in $s\mathscr{S}^+$ with $s'_n \uparrow g$, then $\lim_n \mu(s'_n) = \mu(g)$. It doesn't matter which one we pick. This is why the first equality in the preceding expression is valid.

The next thing we need to check is that if $f, g \in m\mathscr{S}^+$, then $\mu(f+g) = \mu(f) + \mu(g)$. The proof is very similar to the last one. Observing that if $s_n \uparrow f$ and $s'_n \uparrow g$, then $s_n + s'_n \uparrow f + g$. Using proposition 7.2.1 again, we get

$$\mu(f+g) = \lim_{n\to\infty} \mu(s_n + s'_n) = \lim_{n\to\infty} [\mu(s_n) + \mu(s'_n)] = \mu(f) + \mu(g)$$

Using the last two results one after another yields M3.

Solution to Exercise 7.31. Let $A_h = \{s \in s\mathscr{S}^+ : 0 \leq s \leq h\}$ for $h \in \{f, g\}$. If $f \leq g$ pointwise on S, then $A_f \subset A_g$. The expression for the integral in (7.14) now implies that $\mu(f) \leq \mu(g)$.

Solution to Exercise 7.32. Let $\hat{\mu}$ be as defined in the exercise and fix $(A_n) \subset \mathscr{S}$. If (A_n) is disjoint, then $\mathbb{1}_{\cup_n A_n} = \sum_n \mathbb{1}_{A_n}$ holds. Using M3 and M5, we obtain

$$\hat{\mu}(\cup_n A_n) = \mu\left(\sum_n \mathbb{1}_{A_n}\right) = \lim_{k\to\infty} \mu\left(\sum_{n\leq k} \mathbb{1}_{A_n}\right) = \lim_{k\to\infty} \sum_{n\leq k} \mu(\mathbb{1}_{A_n}) = \sum_n \hat{\mu}(A_n)$$

Hence countable additivity holds. The property $\hat{\mu}(\emptyset) = 0$ follows directly from M1.

Solution to Exercise 7.33. Let $(E_n) \subset \mathscr{S}$ have the stated properties. If $f := \mathbb{1}_{\cup_n E_n}$ and $f_n := \mathbb{1}_{E_n}$, then $f_n \uparrow f$ pointwise on S. Hence, by M5, $\mu(f) = \lim_{n\to\infty} \mu(f_n)$. In view of M1, this becomes $\mu(\cup_n E_n) = \lim_{n\to\infty} \mu(E_n)$, which is what we need to show.

Solution to Exercise 7.34. Regarding the first statement, we have $f = f\mathbb{1}_E + f\mathbb{1}_{E^c}$. Hence $\mu(f) = \mu(f\mathbb{1}_E) + \mu(f\mathbb{1}_{E^c})$. Since $\mu(E) = 0$, part 2 of theorem 7.3.5 gives $\mu(f) = \mu(f\mathbb{1}_{E^c})$.

Regarding the second statement, fix $f, g \in \mathscr{L}_1(\mu)$ with $f = g$ μ-a.e. Let E be the set on which f and g disagree. Then, since $\mu(E) = 0$ and $f = g$ on E^c,

$$\mu(f - g) = \mu(\mathbb{1}_E(f-g)) + \mu(\mathbb{1}_{E^c}(f-g)) = 0.$$

Hence $\mu(f) = \mu(g)$.

Solution to Exercise 7.35. The claim is that $f \leq g$ μ-a.e. implies $\mu(f) \leq \mu(g)$. So suppose $f \leq g$ μ-a.e. Then, using $f = f^+ - f^-$ and $g = g^+ - g^-$, we have

$$\mathbb{1}_{E^c} f^+ + \mathbb{1}_{E^c} g^- \leq \mathbb{1}_{E^c} g^+ + \mathbb{1}_{E^c} f^- \tag{7.16}$$

everywhere on S, where E is all x such that $f(x) > g(x)$. We now have ordered nonnegative functions, so, applying M3 of theorem 7.3.5 combined with additivity (M1) yields

$$\mu(\mathbb{1}_{E^c} f^+) + \mu(\mathbb{1}_{E^c} g^-) \leq \mu(\mathbb{1}_{E^c} g^+) + \mu(\mathbb{1}_{E^c} f^-), \tag{7.17}$$

Rearranging gives $\mu(\mathbb{1}_{E^c} f) \leq \mu(\mathbb{1}_{E^c} g)$. Since E has measure zero, $\mu(f) \leq \mu(g)$.

Solution to Exercise 7.36. We need to show that $|f| \in \mathscr{L}_1(\mu)$ and $|\mu(f)| \leq \mu(|f|)$. The first part follows from $|f| = f^+ + f^-$ and the definition of $\mathscr{L}_1(\mu)$, which requires $\mu(f^+) < \infty$ and $\mu(f^-) < \infty$. For the second claim, we have

$$|\mu(f)| = |\mu(f^+ - f^-)| = |\mu(f^+) - \mu(f^-)| \leq \mu(f^+) + \mu(f^-) = \mu(f^+ + f^-) = \mu(|f|)$$

Solution to Exercise 7.37. To see that $(\mu \circ T^{-1})(\emptyset) = 0$, just observe that, for any transformation T, we have $T^{-1}(\emptyset) = \emptyset$. (Since T is a function, each point in the domain has to be mapped to *some* point in S'.)

Regarding countable additivity, let $(A_n) \subset \mathscr{S}'$ be disjoint and let $B_n = T^{-1}(A_n)$. By lemma A.1.1 on page 323, we have $T^{-1}(\cup_n A_n) = \cup_n T^{-1}(A_n) = \cup_n B_n$. Since T is a function and $(A_n) \subset \mathscr{S}'$ is disjoint, the sequence (B_n) is also disjoint. Hence $\mu(\cup_n B_n) = \sum_n \mu(B_n)$. That is,

$$\mu(T^{-1}(\cup_n A_n)) = \mu(\cup_n B_n) = \sum_n \mu(B_n) = \sum_n \mu(T^{-1}(A_n))$$

Put differently, $(\mu \circ T^{-1})(\cup_n A_n) = \sum_n (\mu \circ T^{-1})(A_n)$, as was to be shown.

Solution to Exercise 7.38. The proof that ρ satisfies the definition of a pseudometric is routine. Distinct points can indeed be at zero distance, since $x = (1, 0)$ and $y = (1, 1)$ obey $\rho(x, y) = 0$.

Solution to Exercise 8.2. Although the functional form for the law of motion is more complex, the solution is conceptually the same as the solution to the previous exercise. Further details are omitted.

Solution to Exercise 8.3. It suffices to show that

$$Y = sf(x)W + (1-\delta)x \text{ and } W \sim \phi \implies Y \sim \phi\left(\frac{y - (1-\delta)x}{sf(x)}\right)\frac{1}{sf(x)}$$

where the right hand side is understood as a density in y. This implication follows from theorem 8.1.3 with $\gamma := (1-\delta)x$ and $\Gamma = sf(x)$.

Solution to Exercise 8.4. We can follow the same reasoning we used for exercise 8.1.4 to obtain

$$p(x,y) = \phi\left(\frac{y}{sA(x)f(x)}\right)\frac{1}{sA(x)f(x)}$$

Solution to Exercise 8.5. The solution is essentially the same as that for Exercise 4.25 on page 81, after replacing sums with integrals.

Solution to Exercise 8.6. We need to show that $\int p(x,y)\psi(x)dx = \psi(y)$ for any given $y \in \mathbb{R}$, where p has the form $p(x,y)dy = N(ax,1)$ and $\psi(dy) = N(0, 1/(1-a^2))$. If we fix $y \in \mathbb{R}$, write out the relevant densities and cancel constants, this is equivalent to showing that

$$\frac{1}{\sqrt{2\pi}}\int \exp\left(-\frac{(y-ax)^2}{2} - \frac{x^2(1-a^2)}{2}\right)dx = \exp\left(-\frac{y^2(1-a^2)}{2}\right)$$

Expanding the squares, the left hand side can be written as

$$\frac{1}{\sqrt{2\pi}}\int \exp\left(\frac{-y^2+2axy-x^2}{2}\right)dx$$
$$= \exp\left(-\frac{y^2(1-a^2)}{2}\right)\frac{1}{\sqrt{2\pi}}\int \exp\left(\frac{-(ay)^2+2axy-x^2}{2}\right)dx$$

Since $-(ay)^2+2axy-x^2 = -(x-ay)^2$, the integral evaluates to $\sqrt{2\pi}$. This completes the proof.

Solution to Exercise 8.7. Suppose that ψ^* is a stationary density for p. Then $\psi^*\mathbf{M}^t = \psi^*$ for all t, which means that $\psi^*(y) = \int p^t(x,y)\psi^*(x)dx$ for all $t \in \mathbb{N}$ and $y \in \mathbb{R}$. Fix $y \in \mathbb{R}$ and note that, for any $t \in \mathbb{N}$ and $x \in \mathbb{R}$, we have $p^t(x,y) \leq 1/\sqrt{2\pi t} \leq 1/\sqrt{2\pi}$. Hence $p^t(x,y)\psi^*(x)$ is dominated by the integrable function $(1/\sqrt{2\pi})\psi^*(x)$. Since $p^t(x,y) \to 0$ as $t \to \infty$ for any given x, the dominated convergence theorem implies that

$$\psi^*(y) = \lim_{t\to\infty}\int p^t(x,y)\psi^*(x)dx = 0$$

Since $y \in \mathbb{R}$ was chosen arbitrarily, we conclude that ψ^* is not a density. Contradiction.

Solution to Exercise 8.8. If $m \neq n$, then $|\phi_n - \phi_m| = \mathbb{1}_{[n,n+1)} + \mathbb{1}_{[m,m+1)}$, due to the fact that the supports of these functions are completely disjoint. Hence $d_1(\phi_n, \phi_m) = 2$, as claimed. As a result, all points in the sequence $(\phi_n)_{n \geq 1}$ are isolated in $D(S)$, and no convergent subsequence exists.

Solution to Exercise 8.9. Fix $\phi \in D(S)$ and $n \in \mathbb{N}$. We have

$$\lambda(\phi) = \lambda(\mathbb{1}_{(0,1/n]}\phi) + \lambda(\mathbb{1}_{(1/n,1)}\phi) = \lambda(\mathbb{1}_{(0,1/n]}\phi) + \lambda(\mathbb{1}_{(1/n,1)}|\phi_n - \phi|)$$

Invoking monotonicity gives

$$\lambda(\phi) \leq \lambda(\mathbb{1}_{(0,1/n]}\phi) + \lambda(|\phi_n - \phi|).$$

The first term converges to zero in n by the dominated convergence theorem. The second converges to zero in n by assumption. Hence $\lambda(\phi) = 0$.

Solution to Exercise 8.10. Fix $x \in \mathbb{R}$. By the triangle inequality and $0 \leq G(x) \leq 1$, we have

$$|g(x)| \leq \{|\alpha_1|(1 - G(x)) + |\beta_1|G(x)\} \, |x| + c$$

with $c = |\alpha_0| + |\beta_0|$. The convex combination of two numbers is less than their maximum, so $|g(x)| \leq \gamma|x| + c$.

Solution to Exercise 9.1. Let G be any open subset of $\mathbb{R}$. Since g is continuous, $g^{-1}(G)$ is open in S, and hence $f^{-1}(g^{-1}(G))$ is in $\mathscr{F}$. Since $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$, the function $g \circ f$ pulls open sets back to measurable sets and is therefore Borel measurable. (We are using lemma 7.2.3 on page 173.)

Solution to Exercise 9.2. We can ignore the measure zero set $\mathbb{1}\{x \neq z\}$ when integrating, so

$$\mathbb{E}f = \int f(x)\mathbb{1}\{x = z\}\delta_z(dx) = f(z) \int \mathbb{1}\{x = z\}\delta_z(dx) = f(z)$$

Solution to Exercise 9.3. These are standard results and details are omitted.

Solution to Exercise 9.4. Let $(\Omega, \mathscr{F}, \mathbb{P}) = (S, \mathscr{S}, \mu)$ and X be as stated, so that $X(s) = s$ for all $s \in S$. For any $B \in \mathscr{S}$, we have $X^{-1}(B) = B \in \mathscr{S}$, so X is certainly measurable. Moreover, $\mathbb{P}\{X \in B\} = \mathbb{P}(B) = \mu(B)$, so X has distribution μ.

Solution to Exercise 9.5. The claim is that $X = H^{-1}$ is a Borel measurable function, where H is a strictly increasing cdf. Since H is strictly increasing, it follows that H^{-1} is itself increasing. (You can verify it in a simple proof by contradiction.) The result now follows from exercise 7.27.

Solution to Exercise 9.6. Since H is increasing it preserves inequalities, which means that

$$\mathbb{P}\{X \leq z\} = \lambda\{x : H^{-1}(x) \leq z\} = \lambda\{x : x \leq H(z)\} = H(z)$$

Solution to Exercise 9.7. Pick any $A, B \in \mathscr{T}$. We have

$$\begin{aligned}\mathbb{P}\{g(X) \in A\} \cap \{h(Y) \in B\} &= \mathbb{P}\{X \in g^{-1}(A)\} \cap \{Y \in h^{-1}(B)\} \\ &= \mathbb{P}\{X \in g^{-1}(A)\} \cdot \mathbb{P}\{Y \in h^{-1}(B)\}\end{aligned}$$

where the second equality is by independence of X and Y. We conclude that $g \circ X$ and $h \circ Y$ are also independent.

Solution to Exercise 9.8. Let $\mu_X := \mathbb{E}X$ and $\mu_Y := \mathbb{E}Y$. To see that independence implies $\mathrm{Cov}(X, Y) = 0$, we note that $X - \mu_X$ and $Y - \mu_Y$ are also independent (see exercise 9.7 on page 215), so

$$\mathbb{E}(X - \mu_X)(Y - \mu_Y) = \mathbb{E}(X - \mu_X)\mathbb{E}(Y - \mu_Y) = 0 \cdot 0 = 0$$

Solution to Exercise 9.9. Clearly

$$\{X_t \notin A, \ \forall t \in \mathbb{N}\} = \cap_{t \in \mathbb{N}}\{X_t \notin A\} \subset \cap_{t \leq T}\{X_t \notin A\}$$

for all $T \in \mathbb{N}$. By monotonicity of $\mathbb{P}$ and independent of the (X_t), we then have

$$\mathbb{P}\{X_t \notin A, \ \forall t \in \mathbb{N}\} \leq \mathbb{P} \cap_{t \leq T} \{X_t \notin A\} = (\mathbb{P}\{X_t \notin A\})^T = (1 - \mu(A))^T$$

Since $\mu(A) > 0$, the sequence $(1 - \mu(A))^T$ converges to zero in T, implying that the probability on the left hand side is zero.

Solution to Exercise 9.10. Let (B_n) be a disjoint sequence of Borel sets. Recall that, for such a sequence, we have $\mathbb{1}_{\cup_n B_n} = \sum_n^\infty \mathbb{1}_{B_n}$. Hence, by linearity of the integral and the monotone convergence theorem,

$$\mu_\phi(\cup_n B_n) = \lambda\left(\sum_n^\infty \mathbb{1}_{B_n}\phi\right) = \sum_n^\infty \lambda(\mathbb{1}_{B_n}\phi) = \sum_n^\infty \mu_\phi(B_n)$$

Solution to Exercise 9.11. If such a ϕ exists, then, by setting $B = \mathbb{1}\{x = a\}$, we get $\int_B \phi(x)dx = \delta_a(B) = 1$. But theorem 7.3.5 tells us that $\lambda(B) = 0$ implies $\int_B \phi(x)dx = 0$. Contradiction.

Solution to Exercise 9.12. Evidently $h \geq 0$ implies $\mathbf{M}h(x) = \int h(y)P(x, dy) \geq 0$ for all $x \in S$. In addition, if $|h| \leq M$, then

$$|\mathbf{M}h(x)| = \left| \int h(y)P(x, dy) \right| \leq \int |h(y)|P(x, dy) \leq M \int P(x, dy) = M$$

Solution to Exercise 9.13. This is easy: For any given x, we have $\mathbf{M}\mathbb{1}_S(x) = \int P(x, dy) = 1 = \mathbb{1}_S(x)$.

Solution to Exercise 9.14. This follows directly from monotonicity of the integral. See, for example, theorem 7.3.5 on page 179.

Solution to Exercise 9.15. This follows easily from linearity of the integral. See theorem 7.3.5 on page 179.

Solution to Exercise 9.16. Fix $x \in S$. Observe that $P(x, B) = \phi\{z \in Z : F(x, z) \in B\}$ is the image measure of ϕ under $z \mapsto F(x, z)$. As a consequence of theorem 7.3.9, integrating measurable $h \colon S \to \mathbb{R}$ with respect to the image measure means integrating $h[F(x, z)]$ with respect to ϕ. This confirms (9.17).

Solution to Exercise 10.1. Let $M \in \mathbb{N}$ satisfy $|r| \leq M$. If (x_n) is any sequence in $\mathbb{R}$ and $\sum_n |x_n|$ converges in $\mathbb{R}$, then so does $\sum_n x_n$. (We say that absolute convergence of the sum implies convergence.) Moreover, for any $\omega \in \Omega$,

$$\sum_{t=0}^{\infty} |\rho^t r_\sigma(X_t(\omega))| \leq \sum_{t=0}^{\infty} \rho^t M = M\frac{1}{1-\rho}.$$

Solution to Exercise 10.2. Set $Y_N := \sum_{t=0}^{N} \rho^t r_\sigma(X_t)$. Observe that $|Y_N| \leq M/(1-\rho)$ where M is an upper bound on $|r|$. Since constant functions are integrable when the measure is finite, we can apply the dominated convergence theorem and linearity of the integral to obtain

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \rho^t r_\sigma(X_t)\right] = \mathbb{E} \lim_{N\to\infty} Y_N = \lim_{N\to\infty} \mathbb{E}Y_N = \sum_{t=0}^{\infty} \rho^t \mathbb{E} r_\sigma(X_t)$$

Solution to Exercise 10.3. Fix $x \in S$. The supremum in (10.3) is well-defined because the set of values $\{v_\sigma(x)\}_{\sigma\in\Sigma}$ is bounded above by $M/(1-\rho)$, where $M \in \mathbb{N}$ obeys $|r| \leq M$.

Solution to Exercise 10.4. The only nontrivial part of this problem is checking that the correspondence Γ defined by $\Gamma(a) = [0, a]$ is continuous. This fact is implied by lemma B.1.1 on page 341.

Solution to Exercise 10.5. Fix $w \in bcS$. Boundedness of Tw follows directly from lemma A.2.18 on page 336, which tells us that linear combinations of bounded functions are bounded. Proving continuity is just a matter of checking that all the conditions of Berge's theorem (page 342) are satisfied. That they are follows from the assumption that $w \in bcS$, the dominated convergence theorem, and the restrictions on the primitives. Full details are omitted.

Solution to Exercise 10.6. The aim is to apply Blackwell's condition. For this we need to check that $T \colon bcS \to bcS$ is monotone and, for all $w \in bcS$ and $\gamma \in \mathbb{R}_+$,

$$T(w + \gamma \mathbb{1}_S) \leq Tw + \rho\gamma\mathbb{1}_S \tag{10.13}$$

That T is monotone has already been established. To verify the inequality (10.13), we observe that, at any $x \in S$ and with fixed $\gamma \in \mathbb{R}_+$,

$$T(w + \gamma\mathbb{1}_S)(x) = \max_{u \in \Gamma(x)} \left\{ r(x, u) + \rho \int w[F(x, u, z)]\phi(dz) + \rho\gamma \right\} = Tw(x) + \rho\gamma$$

Hence (10.13) holds, and T is uniformly contracting on (bcS, d_∞) with modulus ρ.

Solution to Exercise 10.7. Let $(P_i)_{i=1}^k$ be a partition of S. Fix $w, v \in b\mathscr{B}(S)$ and $x \in S$. We have

$$|Mv(x) - Mw(x)| = \left| \sum_{i=1}^k v(x_i)\mathbb{1}_{P_i}(x) - \sum_{i=1}^k w(x_i)\mathbb{1}_{P_i}(x) \right|$$

Applying the triangle inequality gives

$$|Mv(x) - Mw(x)| \leq \sum_{i=1}^k |v(x_i) - w(x_i)|\mathbb{1}_{P_i}(x) \leq \sup_{1 \leq i \leq k} |v(x_i) - w(x_i)|$$

(The last inequality uses the fact that partitions are disjoint.) Nonexpansiveness follows directly, since $\sup_{1 \leq i \leq k} |v(x_i) - w(x_i)| \leq d(v, w)_\infty$.

Solution to Exercise 10.8. Fix $w, v \in b\mathscr{B}(S)$ and $x \in S$. Choose i such that $x \in [x_i, x_{i+1}]$. We have

$$|Nw(x) - Nv(x)| = |\lambda(x)(w(x_i) - v(x_i)) + (1 - \lambda(x))(w(x_{i+1}) - v(x_{i+1}))|$$

Since convex combinations are less than suprema, we then have

$$|Nv(x) - Nw(x)| \leq \sup_{1 \leq j \leq k} |v(x_j) - w(x_j)|$$

Nonexpansiveness follows directly.

Solution to Exercise 11.1. To confirm that $X_n \to 0$ almost surely, it suffices to show that $X_n(\omega) \to 0$ for all ω in $(0, 1)$. But this is certainly true, since any $\omega \in (0, 1)$ satisfies $1/n \geq \omega$ for sufficiently large n. The expectation of X_n is $n^2 \cdot (1/n) = n$, which converges to $+\infty$.

Solution to Exercise 11.2. Linear combinations of real-valued Borel measurable functions are Borel measurable. Hence $X_n - X$ is Borel measurable. Continuous transformations of Borel measurable functions are Borel measurable, so $|X_n - X|$ is also Borel measurable. Hence $\{|X_n - X| \geq \epsilon\} \in \mathscr{F}$ for all $\epsilon < 0$, as required.

Solution to Exercise 11.3. Let $(X_t)_{t \geq 1}$ be a zero mean sequence satisfying the stated conditions. Since each X_i is zero mean, so is $\bar{X}_n$. Applying (11.1) on page 251, we have

$$\text{Var}(\bar{X}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E} X_i X_j = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, X_j)$$

Since $\text{Cov}(X_i, X_j) = 0$ for all $i \neq j$ and $\text{Cov}(X_i, X_i) \leq M$ for all i, the double sum above is bounded by $(1/n^2)nM = M/n$.

Solution to Exercise 11.4. Fix $\epsilon > 0$. By the Chebychev inequality (page 214) and exercise 11.3, we have $\mathbb{P}\{|\bar{X}_n| \geq \epsilon\} \leq M/(n\epsilon^2)$. Now take $n \to \infty$.

Solution to Exercise 11.5. It is easy to verify that if T is a uniform contraction with modulus γ and fixed point x^* on metric space (U, ρ), then for any given $x \in U$ we have $\rho(T^k x, x^*) \leq \gamma^k \rho(x, x^*)$. Applying this to the Markov operator $\mathbf{M}$ associated with p, along with theorem 4.3.4 on page 92, we have

$$\left\| p^k(x, dy) - \psi^*(dy) \right\|_1 \leq \gamma^k \|\delta_x - \psi^*\|_1$$

for all $x \in S$, where $\gamma := 1 - \alpha(p)$. Using the definition of the norm and the fact that the norm on the right is bounded by 2 yields the statement in exercise 11.5.

Solution to Exercise 11.6. Since S is finite there exists an $H \in \mathbb{N}$ with $|h| \leq H$. The definition of m and the triangle inequality give

$$\begin{aligned}\left|\sum_{y \in S} h(y) p^k(x, y) - m\right| &= \left|\sum_{y \in S} h(y) p^k(x, y) - \sum_{y \in S} h(y) \psi^*(y)\right| \\ &\leq \sum_{y \in S} |h(y)| \left|p^k(x, y) - \psi^*(y)\right|\end{aligned}$$

Combining with $|h| \leq H$ and the result in exercise 11.5 completes the proof.

Solution to Exercise 11.7. Let $L \in \mathbb{N}$ be such that $|h(x) - m| \leq L$ for all $x \in S$. Using the computations just above exercise 11.7, we have

$$|\operatorname{Cov}(h(X_i), h(X_{i+k}))| \leq \sum_{x \in S} |h(x) - m| \psi^*(x) \left| \sum_{y \in S} [h(y) - m] p^k(x, y) \right|$$
$$\leq L \sum_{x \in S} \left| \sum_{y \in S} h(y) p^k(x, y) - m \right| \psi^*(x)$$

From the result in exercise 11.6, the right hand side is bounded by $L \sum_{x \in S} K\gamma^k \psi^*(x) = LK\gamma^k$, where $\gamma \in [0, 1)$. This verifies the claim in exercise 11.7.

Solution to Exercise 11.8. We just need to check the two conditions of theorem 11.1.7 for the process $(Y_t) := (h(X_t))$. The bound on the covariance terms follows directly from exercise 11.7. We also require that $\mathbb{E}h(X_t)$ converges to some constant. However, we assumed that (X_t) is stationary, with $\mathbb{E}h(X_t) = m$ for all t. So this convergence is trivial. Hence all the conditions of the theorem are verified.

Solution to Exercise 11.9. Fix $\mu \in b\mathscr{M}(S)$. We have $S = S \cup \emptyset$ and the union is disjoint, so $\mu(S) = \mu(S) + \mu(\emptyset)$. That $\mu(\emptyset) = 0$ now follows from finiteness of $\mu(S)$, which is part of the definition of a signed measure.

Solution to Exercise 11.11. For both claims, we discuss only μ^+, since the case of μ^- is similar. Regarding the first claim, we need only show that μ^+ is nonnegative and countably additive. Nonnegativity is obvious. For countable additivity, take (B_n) to be a disjoint sequence in $\mathscr{B}(S)$. Since $(B_n \cap S^+)$ is also disjoint, we have

$$\mu^+(\cup_n B_n) = \mu((\cup_n B_n) \cap S^+) = \mu(\cup_n (B_n \cap S^+)) = \sum_n \mu(B_n \cap S^+) = \sum_n \mu^+(B_n)$$

Regarding the claim $\mu(S^+) = \max_{B \in \mathscr{B}(S)} \mu(B)$, for any $B \in \mathscr{B}(S)$, we have

$$\mu(B) = \mu(B \cap S^+) + \mu(B \cap S^-) \leq \mu(B \cap S^+) \leq \mu(S^+)$$

where the last inequality is by monotonicity of μ restricted to S^+.

Solution to Exercise 11.12. Fix $f \in m\mathscr{B}(S)$ with $\lambda(|f|) < \infty$ and let $\mu(B) := \lambda(\mathbb{1}_B f)$. To verify that $\mu \in b\mathscr{M}(S)$, we only need to check countable additivity. So let $(B_n) \subset \mathscr{B}(S)$ be disjoint and recall that, for such a sequence, $\mathbb{1}_{\cup_n B_n} = \sum_n \mathbb{1}_{B_n}$. Hence, by additivity of λ and the dominated convergence theorem,

$$\mu(\cup_n B_n) = \lambda(\sum_n \mathbb{1}_{B_n} f) = \sum_n \lambda(\mathbb{1}_{B_n} f) = \sum_n \mu(B_n)$$

To see that S^+ and S^- form a Hahn decomposition of S with respect to μ, we need only verify that they form a measurable partition of S with $\mu(B) \geq 0$ for measurable $B \subset S^+$ and $\mu(B) \leq 0$ for measurable $B \subset S^-$. All of these results are obvious from the definitions $S^+ = \{x \in S : f(x) \geq 0\}$ and $S^- = \{x \in S : f(x) < 0\}$.

In addition, $\mu^+(B) = \lambda(\mathbb{1}_B f^+)$ holds because

$$\mu^+(B) = \mu(B \cap S^+) = \mu(B \cap \{f \geq 0\}) = \lambda(\mathbb{1}_B \mathbb{1}\{f \geq 0\} f) = \lambda(\mathbb{1}_B f^+)$$

The proof for μ^- is similar. Finally,

$$\|f\|_1 = \lambda(|f|) = \lambda(f^+) + \lambda(f^-) = \lambda(\mathbb{1}_{S^+} f) + \lambda(\mathbb{1}_{S^-} f) = \mu(S^+) + \mu(S^-)$$

as was to be shown.

Solution to Exercise 11.13. Fix $\mu \in b\mathscr{M}(S)$. Let $M = \max_{\pi \in \Pi} \sum_{A \in \pi} |\mu(A)|$. Let $\hat{\pi} = \{S^+, S^-\}$, where S^+ and S^- are as in theorem 11.1.9. Then $\hat{\pi}$ is in Π and, moreover,

$$\|\mu\|_{TV} = \mu(S^+) - \mu(S^-) = |\mu(S^+)| + |\mu(S^-)| = \sum_{A \in \hat{\pi}} |\mu(A)| \leq M$$

Moreover, for other $\pi \in \Pi$, we have

$$\sum_{A \in \pi} |\mu(A)| \leq \sum_{A \in \pi} \mu(A \cap S^+) - \sum_{A \in \pi} \mu(A \cap S^-) = \mu(S^+) - \mu(S^-) = \|\mu\|_{TV}$$

Hence $M \leq \|\mu\|_{TV}$. We conclude that $M = \|\mu\|_{TV}$, as was to be shown.

Solution to Exercise 11.14. Let $\|\cdot\| := \|\cdot\|_{TV}$. As you can easily verify, it suffices to show that $\|\cdot\|$ has all the properties of a norm on $b\mathscr{M}$. In particular, we need to show that, for any $\mu, \nu \in b\mathscr{M}$, we have (a) $\|\mu\| = 0$ iff $\mu = 0$, (b) $\|\alpha\mu\| = |\alpha|\|\mu\|$ for all $\alpha \in \mathbb{R}$ and (c) $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$.

For (a), that $\|\mu\| = 0$ when $\mu = 0$ is clear from $\|\mu\| = \max_{\pi \in \Pi} \sum_{A \in \pi} |\mu(A)|$. To see that the reverse implication holds, suppose μ is not the zero measure. Then there exists a $B \in \mathscr{B}(S)$ with $|\mu(B)| > 0$. Hence

$$\|\mu\| = \max_{\pi \in \Pi} \sum_{A \in \pi} |\mu(A)| \geq |\mu(B)| + |\mu(B^c)| > 0.$$

Part (b) follows from

$$\|\alpha\mu\| = \max_{\pi \in \Pi} \sum_{A \in \pi} |\alpha\mu(A)| = \max_{\pi \in \Pi} \sum_{A \in \pi} |\alpha||\mu(A)|$$

Regarding part (c), we have

$$\|\mu + \nu\| = \max_{\pi \in \Pi} \sum_{A \in \pi} |\mu(A) + \nu(A)| \leq \max_{\pi \in \Pi} \sum_{A \in \pi} |\mu(A)| + \max_{\pi \in \Pi} \sum_{A \in \pi} |\nu(A)|$$

The proof is now done.

Solution to Exercise 11.15. For each $n \in \mathbb{N}$ we have

$$\sup_{B \in \mathscr{B}(S)} |\phi_n(B) - \phi(B)| \geq |\phi_n((0,\infty)) - \phi(((0,\infty))| = |\phi_n((0,\infty))| = 1$$

From this fact, combined with lemma 11.1.13, we see that $d_{TV}(\phi_n, \phi) \to 0$ fails.

Solution to Exercise 11.16. Let $\phi_n := \delta_{1/n}$ and $\phi := \delta_0$. For fixed $h \in bcS$, we have $\phi_n(h) = h(1/n) \to h(0) = \phi(h)$. Hence $\phi_n \to \phi$ weakly.

Solution to Exercise 11.17. We give a counterexample to the claim that convergence in distribution implies convergence in probability. Suppose (X_n) is IID and binary, hitting -1 and 1 with equal probability. The distribution sequence is constant and therefore convergence in distribution holds. Now suppose there exists a Z such that $X_n \to Z$ in probability. Fix $\epsilon > 0$. Note that $|X_n - X_m| \leq |X_n - Z| + |X_m - Z|$, so

$$|X_n - X_m| > \epsilon \implies |X_n - Z| > \epsilon/2 \text{ or } |X_m - Z| > \epsilon/2$$

Therefore, since $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ for all A, B,

$$\mathbb{P}\{|X_n - X_m| > \epsilon\} \leq \mathbb{P}\{|X_n - Z| > \epsilon/2\} + \mathbb{P}\{|X_m - Z| > \epsilon/2\}$$

Hence $\mathbb{P}\{|X_n - X_{n+1}| > \epsilon\} \to 0$ as $n \to \infty$. But X_n and X_{n+1} are independent, so, for small ϵ,

$$\mathbb{P}\{|X_n - X_{n+1}| > \epsilon\} \geq \mathbb{P}\{X_n = -1 \text{ and } X_{n+1} = 1\} = \frac{1}{4}$$

Contradiction.

Solution to Exercise 11.18. Suppose $\phi_n \to \phi$ and $\phi_n \to \phi'$, where ϕ and ϕ' are elements of $\mathscr{P}(S)$. Then, for any $h \in bcS$, we have $\phi(h) = \lim_n \phi_n(h) = \phi'(h)$. But then $\phi = \phi'$, by part 2 of theorem 11.1.16,

Solution to Exercise 11.19. For this model, we have $P(x, B) = \mathbb{1}_B(x)$. Given any distribution $\psi \in \mathscr{P}(S)$ and Borel set B,

$$\int P(x, B)\psi(dx) = \int \mathbb{1}_B(x)\psi(dx) = \psi(B)$$

Hence ψ is stationary for P.

Solution to Exercise 11.20. Let $G(x) = Ax + b$, where A and b are as in example 11.2.10. Clearly G is continuous. We also need to show that there exists an $M < \infty$ and $\alpha < 1$ such that $\|G(x)\| \leq \alpha\|x\|$ whenever $\|x\| > M$. To see that this is so, note that, as discussed in example 11.2.10, $\|G(x)\| = \|Ax + b\| \leq \lambda\|x\| + \|b\|$. We then have

$$\frac{\|G(x)\|}{\|x\|} \leq \lambda + \frac{\|b\|}{\|x\|}$$

Now choose $\alpha \in (\lambda, 1)$. Since $\|b\|/\|x\| \to 0$ as $\|x\| \to \infty$, we will have $\|G(x)\|/\|x\| \leq \alpha$ when $\|x\|$ is sufficiently large (more precisely, when $\|x\| \geq \|b\|/(\alpha - \lambda)$).

Solution to Exercise 11.21. The only challenge is to show existence of a norm-like function $w\colon S := (0, \infty) \to \mathbb{R}_+$ and constants $\alpha \in [0, 1)$ and $\beta \in \mathbb{R}_+$ with $\int w[sk^\alpha z]\phi(dz) \leq \alpha w(k) + \beta$ for all $k \in S$. For this purpose we take $w(k) := |\ln k|$. We saw in lemma 8.2.12 that this function is norm-like on S. Moreover,

$$\int w[sk^\alpha z]\phi(dz) = \int |\ln s + \alpha \ln k + \ln z|\phi(dz) \leq \alpha \ln |k| + |\ln s| + \int |\ln z|\phi(dz)$$

We now have the desired bound with $\beta := |\ln s| + \int |\ln z|\phi(dz)$.

Solution to Exercise 11.22. Let μ have density f and ν have density g. The claim is that $\mu \wedge \nu$ has density $f \wedge g$. To see that this is so, we need to show that $\eta := \mu \wedge \nu$ obeys $\eta(B) = \lambda(\mathbb{1}_B f \wedge g)$ for all $B \in \mathscr{B}(S)$. Fixing such a B, we easily see that $\eta(B) \leq \lambda(\mathbb{1}_B f) = \mu(B)$ and similarly for ν. Hence $\eta \leq \mu$ and $\eta \leq \nu$. All that remains to be shown is that, for any $\kappa \in b\mathscr{M}(S)$ with $\kappa \leq \mu$ and $\kappa \leq \nu$ we have $\kappa \leq \eta$. But this is also clear, since

$$\kappa(B) \leq \lambda(\mathbb{1}_B f) \text{ and } \kappa(B) \leq \lambda(\mathbb{1}_B g) \implies \kappa(B) \leq \lambda(\mathbb{1}_B f \wedge g)$$

Solution to Exercise 11.23. Fix μ and ν in $\mathscr{P}(S)$. Set $M := \min_{\pi \in \Pi} \sum_{A \in \pi} \mu(A) \wedge \nu(A)$ and $\hat{\pi} = \{S^+, S^-\}$, where S^+ and S^- are the positive and negative set for $\mu - \nu$ used in the proof of lemma 11.2.14 on page 265. By construction, $\mu(B) \leq \nu(B)$ for $B \in S^-$ and $\mu(B) \geq \nu(B)$ for $B \in S^+$. Since $\hat{\pi}$ is a measurable partition, we have

$$M \leq \sum_{A \in \hat{\pi}} \mu(A) \wedge \nu(A) = (\mu \wedge \nu)(S^-) + (\mu \wedge \nu)(S^+) = \mu(S^-) + \nu(S^+) = \operatorname{aff}(\mu, \nu)$$

At the same time, for any $\pi \in \Pi$,

$$\operatorname{aff}(\mu, \nu) = \sum_{A \in \pi} (\mu \wedge \nu)(A) \leq \sum_{A \in \pi} \mu(A) \wedge \nu(A)$$

so $\operatorname{aff}(\mu, \nu) \leq M$ also holds. Hence $\operatorname{aff}(\mu, \nu) = M$, as claimed.

Regarding the second part of the question, clearly

$$\operatorname{aff}(\mu, \nu) = (\mu \wedge \nu)(S) \leq \mu(S) = 1$$

Moreover, if $\mu = \nu$, then, since every $\pi \in \Pi$ is a measurable partition.

$$\operatorname{aff}(\mu, \nu) = \min_{\pi \in \Pi} \sum_{A \in \pi} \mu(A) \wedge \mu(A) = \min_{\pi \in \Pi} \sum_{A \in \pi} \mu(A) = 1$$

Finally, if μ and ν are distinct, then there exists a $B \in \mathscr{B}(S)$ with $\mu(B) < \nu(B)$. As a result,

$$\text{aff}(\mu, \nu) = \min_{\pi \in \Pi} \sum_{A \in \pi} \mu(A) \wedge \nu(A) \leq \mu(B) \wedge \nu(B) + \mu(B^c) \wedge \nu(B^c) < \nu(B) + \nu(B^c) = 1$$

Solution to Exercise 11.24. Suppose (11.15) holds and fix $x, x' \in S$. We have $P_x^m \geq \epsilon\nu$ and $P_{x'}^m \geq \epsilon\nu$, so $P_x^m \wedge P_{x'}^m \geq \epsilon\nu$. Evaluating at S gives $\text{aff}(P_x^m, P_{x'}^m) \geq \epsilon$. Hence $\alpha(P^m) \geq \epsilon > 0$.

Solution to Exercise 11.25. Suppose condition M holds for some $m \in \mathbb{N}$ and $\epsilon > 0$. Fix $x, x' \in S$. We have $(P_x^m \wedge P_{x'}^m)(S) = P^m(x, S^-) + P^m(x', S^+)$, where S^- and S^+ are negative and positive for $P^m(x, dy) - P^m(x', dy)$ respectively. In addition, $S^+ = (S^-)^c$. Hence, by condition M,

$$\text{aff}(P_x^m, P_{x'}^m) = (P_x^m \wedge P_{x'}^m)(S) = P^m(x, S^-) + P^m(x', (S^-)^c) \geq \epsilon > 0.$$

As a result, $\alpha(P^m) > 0$.

Solution to Exercise 11.28. If the SRS is monotone increasing and $h \in ibS$, then $x \leq x'$ implies $h[F(x,z)] \leq h[F(x',z)]$ for all $z \in Z$, so, by monotonicity of the integral,

$$\mathbf{M}h(x) = \int h[F(x,z)]\phi(dz) \leq \int h[F(x',z)]\phi(dz) = \mathbf{M}h(x')$$

Hence $\mathbf{M}h \in ibS$, as was to be shown.

Solution to Exercise 11.29. Let $B \in \mathscr{B}(S)$ be an increasing set. The function $\mathbb{1}_B$ is bounded and Borel measurable. In addition, with $x \leq x'$, we have $x \in B$ implies $x' \in B$ and hence $\mathbb{1}_B(x) \leq \mathbb{1}_B(x')$. The reverse implication follows from similar logic.

Solution to Exercise 11.30. Let B be an increasing set and let the SRS be monotone increasing. Fix $m \in \mathbb{N}$. In view of exercise 11.29, the function $\mathbf{M}\mathbb{1}_B$ is increasing. Applying $\mathbf{M}$ to this function proves that $\mathbf{M}^2\mathbb{1}_B$ is increasing and so on up to $\mathbf{M}^m\mathbb{1}_B$. But $P^m(x, B) = \mathbf{M}^m\mathbb{1}_B(x)$, so $x \mapsto P^m(x, B)$ is increasing as required.

Solution to Exercise 11.31. Let $\psi^{**} \in \mathscr{P}(S)$ satisfy $\psi^{**}\mathbf{M} = \psi^{**}$ and suppose that (11.31) holds. Fix $h \in ibS$. We then have

$$\psi^{**}(h) = (\psi^{**}\mathbf{M}^t)(h) \to \psi^*(h)$$

From this argument we see that $\psi^{**}(h) = \psi^*(h)$ for all $h \in ibS$. Applying theorem 11.1.16 on page 257 now gives $\psi^{**} = \psi^*$.

Solution to Exercise 11.32. The claim in exercise 11.32 holds because $N_j := \cup_{t \leq j}\{X_t \leq X'_t\}$ is increasing in the sense of set inclusion: if the paths have become ordered some time prior to j, then they have become ordered some time prior to $j+1$. Hence, by exercise 7.17 on page 165, we have

$$\mathbb{P} \cup_{t \geq 0} \{X_t \leq X'_t\} = \lim_{j \to \infty} \mathbb{P} \cup_{t \leq j} \{X_t \leq X'_t\} = 1 - \lim_{j \to \infty} \mathbb{P} \cap_{t \leq j} \{X_t \nleq X'_t\}$$

Solution to Exercise 11.33. We prove only the first inequality. Since $a \leq c \leq b$, the set $[c, b]$ is an increasing subset of $S = [a, b]$, so, by exercise 11.30 and the fact that the SRS is monotone increasing, the function $x \mapsto P^m(x, [b, c])$ is increasing. As a consequence,

$$P^m(x, [c, b]) \geq P^m(a, [c, b]) \geq \epsilon$$

for all $x \in S$.

(If you wish to check the second inequality, you can introduce the notion of a decreasing set, defined analogously to an increasing set, and then show that (i) the interval $[a, c]$ is decreasing in S and (ii) the function $x \mapsto P^m(x, B)$ is decreasing whenever B is decreasing.)

Solution to Exercise 11.34. The first claim is that all measurable subsets of order inducing sets are order inducing. This is quite obvious because infima over smaller sets are larger. So if, say, $\inf_{x \in C} P^m(x, \{z : z \leq c\}) > 0$, then the same is true when we take the infimum over $C' \subset C$.

The second claim follows from the first. It says that, to check the order norm-like property, we only need to check that sufficiently large sublevel sets are ordering inducing. This is true because smaller sublevel sets are contained in these larger ones, and hence are automatically order inducing.

Solution to Exercise 11.35. If $v(x) := 1/x + x$, then sublevel sets of v are closed intervals in S. Hence, by the argument immediately above the exercise, the function v is order norm-like on S.

Solution to Exercise 11.36. Fix any constant $\alpha_1 \in (0, 1)$. Since $\lim_{x \to \infty} f(x)/x = 0$, we can choose a $\gamma \in S$ satisfying

$$sf(x)\mathbb{E}W_1 \leq \alpha_1 x \qquad \forall\, x > \gamma$$

Given monotonicity of f, we can take a $\beta_1 \in \mathbb{R}_+$ with

$$sf(x)\mathbb{E}W_1 \leq \beta_1 \qquad \forall\, x \leq \gamma$$

Combining these two inequalities, we get

$$\mathbf{M}v_1(x) = sf(x)\mathbb{E}W_1 \leq \alpha_1 x + \beta_1 = \alpha_1 v_1(x) + \beta_1 \qquad \forall\, x \in S$$

Solution to Exercise 11.37. Fix any constant $\alpha_2 \in (0,1)$. Since $\lim_{x \to 0} f(x)/x = \infty$, we can obtain a $\gamma \in S$ satisfying

$$\mathbb{E}\left[\frac{1}{sf(x)W_1}\right] \leq \alpha_2 \frac{1}{x} \qquad \forall\, x < \gamma$$

Using monotonicity of f, we can also choose a $\beta_2 \in \mathbb{R}_+$ with

$$\mathbb{E}\left[\frac{1}{sf(x)W_1}\right] \leq \beta_2 \qquad \forall\, x \geq \gamma$$

Combining these two inequalities, we get

$$\mathbf{M}v_2(x) = \mathbb{E}\left[\frac{1}{sf(x)W_1}\right] \leq \alpha_2 \frac{1}{x} + \beta_2 = \alpha_2 v_2(x) + \beta_2 \qquad \forall\, x \in S$$

Solution to Exercise 11.38. This is straightforward: Fix $x, x' \in C$. By the (ν, ϵ)-small property, we have $P_x \geq \epsilon\nu$ and $P_{x'} \geq \epsilon\nu$. As a consequence, by the definition of the infimum, $P_x \wedge P_{x'} \geq \epsilon\nu$. Evaluating at S yields $\gamma(x, x') \geq \epsilon$, as claimed.

Solution to Exercise 11.39. The claim is that $C' \subset C$ is small whenever C is small and C' is measurable. This is obvious: If the bound $P(x, A) \geq \epsilon\nu(A)$ is true for all $x \in C$, then certainly it is true for any $x \in C' \subset C$.

Solution to Exercise 11.40. Let $P(x, dy) = p(x, y)dy$. Let g have the stated property (g is nonnegative, measurable, $\int g(y)dy > 0$ and $p(x, y) \geq g(y)$ for all $x \in C$ and $y \in S$). Fix $x \in C$ and $A \in \mathscr{B}(S)$. We have $\int_A p(x, y)dy \geq \eta(A)$ when η is the Borel measure given by $\eta(B) := \int_B g(y)dy$. Set $\epsilon := \eta(S) = \int g(y)dy > 0$ and $\nu(A) = \eta(A)/\epsilon$. Then $P(x, A) \geq \epsilon\nu(A)$. Hence C is small for P.

Solution to Exercise 11.41. It suffices to show that, for all $b \in \mathbb{R}$, the interval $C := [-b, b]$ is small for this kernel, since every bounded measurable set lies in such an interval. We will only use the fact that p is everywhere positive and continuous on $\mathbb{R} \times \mathbb{R}$, which in turn implies the existence of a constant $r > 0$ such that $p(x, y) \geq r$ whenever $-b \leq x, y \leq b$. Now set $g = r\mathbb{1}_{[-b,b]}$. For $x \in C$, we have $p(x, y) \geq r\mathbb{1}_{[-b,b]} = g$. Applying exercise 11.40, we see that C is small for P.

Solution to Exercise 11.42. Assume the conditions of lemma 11.3.16. We can also assume, without loss of generality, that v in the lemma satisfies $v \geq 1$. (It is not difficult to confirm that if the lemma holds for some v, α and β then it also holds for the function $v' := v + 1$ and constants $\alpha' := \alpha$ and $\beta' = \beta + 1$.) Now pick any $\lambda \in (\alpha, 1)$, set $C := \{x : v(x) \leq \beta/(\lambda - \alpha)\}$ and $L := \beta$. Note that C, a sublevel set, is small (by assumption). We claim that then v, C, λ, and L satisfy the conditions of definition 11.3.15. To see this, we first take $x \in C$. Then $\mathbf{M}v(x) \leq \alpha v(x) + \beta \leq \lambda v(x) + L\mathbb{1}_C(x)$ At the same time, if $x \notin C$, then $v(x) > \beta/(\lambda - \alpha)$, so

$$\frac{\mathbf{M}v(x)}{v(x)} \leq \alpha + \frac{\beta}{v(x)} \leq \alpha + (\lambda - \alpha) = \lambda.$$

Hence, in both cases, $\mathbf{M}v(x) \leq \lambda v(x) + L\mathbb{1}_C(x)$.

Solution to Exercise 11.43. Recall from the solution to exercise 11.40 that, under the stated conditions, C is (ϵ, ν)-small for P with $\epsilon := \int g(y)dy > 0$ and ν defined by $\nu(A) = \int_A g(y)dy/\epsilon$. Since $\int_C g(x)dx = \epsilon\nu(C)$, it is clear that $\int_C g(x)dx > 0$ implies $\nu(C) > 0$. Hence P is aperiodic.

Solution to Exercise 11.44. Recall that the kernel P for the STAR model satisfies $P(x, dy) = p(x,y)dy$ where $p(x,y) = \phi(y - g(x))$. Fix $f \in D(S)$ and let $\mu \in \mathscr{P}(S)$ be defined by $\mu(B) = \int_B f(x)dx$. Since ϕ is everywhere positive, for any $x \in S$ and $B \in \mathscr{B}(S)$ with positive Lebesgue measure, we have $P(x, B) = \int_B \phi(y - g(x))dy > 0$. If $\mu(B) > 0$, then $\lambda(B) > 0$, so $P(x, B) > 0$. (Integrals of positive functions over sets of positive measure have positive value.) Hence P is μ-irreducible.

Solution to Exercise 11.45. We need to show (a) that $p(x,y) \geq g(y)$ for all $x \in C$ and $y \in \mathbb{R}$, and (b) that $\int_C g(x)dx > 0$. Regarding (a), fix $x \in C$. If $y \in C$, then $p(x,y) \geq \delta = \delta\mathbb{1}_C(y) =: g(y)$ by definition of δ. If $y \notin C$, then $g(y) = 0$, so the inequality is trivial. Hence (a) holds. Regarding (b), recalling that $\lambda(C) > 0$, we have $\delta \int_C \mathbb{1}_C(x)dx = \delta\lambda(C) > 0$. The proof is now done.

Solution to Exercise 12.1. Recall that $f_n \to f$ uniformly implies $f_n \to f$ pointwise. Hence, if $(f_n) \subset ibcS$ and $f_n \to f \in bcS$ uniformly, then f is increasing. Indeed, $x \leq x'$ implies $f_n(x) \leq f_n(x')$ for all n and limits in $\mathbb{R}$ preserve weak inequalities. Hence $f(x) \leq f(x')$. The same is not true for the set of strictly increasing functions because limits to not in general preserve strict inequalities.

Solution to Exercise 12.2. Under the stated hypothesis, the weak inequality $Tw(x) \leq Tw(x')$ in the proof of theorem 12.1.1 can be strengthened to $Tw(x) < Tw(x')$. Hence T sends $ibcS$ into the *strictly* increasing functions in $ibcS$. Since $v^* \in ibcS$, as established by theorem 12.1.1, it follows that Tv^* is strictly increasing. But then v^* is strictly increasing, since $v^* = Tv^*$.

Solution to Exercise 12.3. This just a matter of checking the conditions of theorem 12.1.1. Clearly $a \leq a'$ implies $\Gamma(a) = [0, a] \subset [0, a'] = \Gamma(a')$. Also, both rewards and the next period state are increasing in the current state. (The transition function is $f(s, z)$, which is a function of s, the action, and not the state. Hence $f(s,z)$ is weakly increasing in the state a.)

Solution to Exercise 12.4. Let g and h satisfy the stated conditions and let $f = g + h$. Fix $x, x' \in S$ with $x < x'$ and $u, u' \in \Gamma(x) \cap \Gamma(x')$ with $u < u'$. We have

$$f(x,u') - f(x,u) = g(x,u') + h(x,u') - g(x,u) - h(x,u) \\ = [g(x,u') - g(x,u)] + [h(x,u') - h(x,u)]$$

Since g has strictly increasing differences and h has increasing differences, the last term is strictly dominated by $g(x',u') - g(x',u) + h(x,u') - h(x,u)$, which equals $f(x',u') - f(x',u)$.

Solution to Exercise 12.5. The correspondence is $\Gamma(a) = [0, a]$, which is a decreasing set in $\mathbb{R}_+$. That rewards have strictly increasing differences under the stated assumptions was proved in 12.1.3. Hence we need only check the last condition of corollary 12.1.5, which is that $(x,u) \mapsto \int w[F(u,x,z)]\phi(dz)$ has increasing differences whenever $w \in ibcS$. In the optimal savings model, this translates to $(x,u) \mapsto \int w[f(u,z)]\phi(dz)$, which certainly has (weakly) increasing differences on $\operatorname{gr}\Gamma$, being independent of x.

Solution to Exercise 12.6. In exercise 12.1, we proved that $ibcS$ is a closed subset of (bcS, d_∞) by invoking the fact that weak inequalities are preserved under pointwise limits. The proof that $\mathscr{C}ibcS$ is a closed subset of $ibcS$ is very similar in spirit and further details are omitted.

Solution to Exercise 12.7. The argument is very similar to that of exercise 12.2. Under the stated condition, T maps elements of $\mathscr{C}ibcS$ into strictly concave elements of $\mathscr{C}ibcS$. Since $v^* \in \mathscr{C}ibcS$ and $Tv^* = v^*$, strict concavity of v^* holds.

Solution to Exercise 12.8. The steps are quite routine by now, given the previous results, and the details are omitted.

Solution to Exercise 12.9. Let $g \colon [a,b] \to \mathbb{R}$ be strictly concave and suppose that x and x' are distinct maximizers in $[a,b]$, with (necessarily) common value $m = g(x) = g(x')$. Then, by strict concavity,

$$g(0.5x + 0.5x') > 0.5g(x) + 0.5g(x') = m.$$

Since $x'' := 0.5x + 0.5x'$ is in $[a,b]$, this contradicts the statement that x and x' are maximizers. Hence g has at most one maximizer, as claimed. Under the conditions of exercise 12.8, the right hand side of the Bellman equation is strictly concave, so there is one and only one optimal policy for the savings model. Continuity now follows from Berge's theorem (page 342).

Solution to Exercise 12.10. Under the conditions of corollary 12.1.10 we have $(v^*)'(a) = U'(a - \sigma(a))$ for all $a > 0$. Since U is strictly concave, U' is strictly decreasing and therefore invertible with strictly decreasing inverse. Denoting that inverse by h, we have $a - \sigma(a) = h((v^*)'(a))$. Because v^* is strictly concave, its derivative is strictly decreasing. Hence $a \mapsto a - \sigma(a)$ is strictly increasing.

Solution to Exercise 12.11. We provide the key ideas of the proof. Fix $a_0 \in (0, \infty)$, set $s_0 := \sigma(a_0)$ and let

$$h(a) := U(a - s_0) + \rho \int v^*[f(s_0, z)]\phi(dz)$$

Then $h(a) \leq v^*(a)$ in a sufficiently small neighborhood of a_0, where s_0 is a feasible choice. (The neighborhood is nonempty because s_0 is interior.) Moreover, $h(a_0) = v^*(a_0)$ holds. Hence the derivative of v^* at a_0 exists and is equal to $h'(a_0)$. By the definition of h, this is $U'(a_0 - s_0) = U'(a_0 - \sigma(a_0))$.

Solution to Exercise 12.12. We have $\mathbf{M}w_i \leq \alpha_i w_i + \beta_i$ pointwise on S for $i = 1, 2$. As a result, by linearity of $\mathbf{M}$,

$$\mathbf{M}w = \mathbf{M}w_1 + \mathbf{M}w_2 \leq \alpha_1 w_1 + \beta_1 + \alpha_2 w_2 + \beta_2 \leq \alpha(w_1 + w_2) + \beta$$

where $\alpha := \max\{\alpha_1, \alpha_2\}$ and $\beta := \beta_1 + \beta_2$.

Solution to Exercise 12.13. This equivalence follows directly from the definition of $b_\kappa S$.

Solution to Exercise 12.14. If $v \in b\mathscr{B}(S)$, then $|v| \leq M$ for some $M \in \mathbb{N}$. But then $|v|/\kappa \leq M$, since $\kappa \leq 1$. Hence $v \in\subset b_\kappa \mathscr{B}(S)$. The proof of the second case is similar.

Solution to Exercise 12.15. The only nontrivial part of the proof is the triangle inequality. This is still quite straightforward: If $u, v, w \in b_\kappa S$, then, using add and subtract followed by the triangle inequality in $\mathbb{R}$,

$$\left|\frac{u}{\kappa} - \frac{w}{\kappa}\right| \leq \left|\frac{u}{\kappa} - \frac{v}{\kappa}\right| + \left|\frac{v}{\kappa} - \frac{w}{\kappa}\right| \leq \|u - v\|_\kappa + \|v - w\|_\kappa$$

Taking the supremum yields $d_\kappa(u, w) \leq d_\kappa(u, v) + d_\kappa(v, w)$, as was to be shown.

Solution to Exercise 12.16. Regarding the first claim, suppose κ is Borel measurable. Pointwise limits of measurable functions are measurable and d_κ convergence implies pointwise convergence, so $b_\kappa \mathscr{B}(S)$ is closed in $(b_\kappa S, d_\kappa)$. Regarding the second claim, suppose κ is continuous and let (v_n) be a sequence in $b_\kappa cS$ converging to $v \in b_\kappa S$. Since $v_n/\kappa \to v/\kappa$ uniformly, and since uniform limits of continuous functions are continuous, the function v/κ is continuous. Products of continuous functions are continuous, so $\kappa(v/\kappa)$ is continuous. That is, $v \in b_\kappa cS$.

Solution to Exercise 12.17. We provide the proof that T is invariant on $b_\kappa cS$. Continuity of Tw follows from lemma 12.2.15, the continuity of r and Berge's theorem of the maximum (page 342). Hence we need only show that Tw is κ-bounded. To verify this, we use lemma A.2.18 on page 336, combined with the triangle inequality, to obtain

$$|Tw(x)| \leq \max_{u \in \Gamma(x)} |r(x,u)| + \max_{u \in \Gamma(x)} \rho \int |w[F(x,u,z)]| \phi(dz)$$

By assumption 12.2.9, the first term is bounded by $R\kappa(x)$. Applying the second part of the same assumption yields

$$\max_{u \in \Gamma(x)} \rho \int |w[F(x,u,z)]| \phi(dz) \leq \max_{u \in \Gamma(x)} \rho \int \|w\|_\kappa \kappa[F(x,u,z)] \phi(dz) \leq \rho \|w\|_\kappa \beta \kappa(x)$$

As a result, $|Tw(x)| \leq R\kappa(x) + \rho\|w\|_\kappa \beta\kappa(x)$. It follows directly that Tw is κ-bounded.

Solution to Exercise 12.18. We claim the existence of a $\lambda < 1$ such that

$$T(v + a\kappa) \leq Tv + \lambda a \kappa \quad \text{for all } v \in b_\kappa cS \text{ and } a \in \mathbb{R}_+ \tag{12.18}$$

To see that this is so, fix $v \in b_\kappa cS$ and $a \in \mathbb{R}_+$. We have

$$\begin{aligned}(T(v + a\kappa))(x) &= \max_{u \in \Gamma(x)} \left\{ r(x,u) + \rho \int (v + a\kappa)[F(x,u,z)] \phi(dz) \right\} \\ &= Tv(x) + \rho \max_{u \in \Gamma(x)} a \int \kappa[F(x,u,z)] \phi(dz)\end{aligned}$$

Applying assumption 12.2.9 leads to the bound

$$(T(v + a\kappa))(x) \leq Tv(x) + a\rho\beta\kappa(x)$$

In this expression, as part of assumption 12.2.9, β can be chosen to satisfy $\beta\rho < 1$. With $\lambda := \beta\rho$, the proof is now complete.