Existence and Uniqueness of Equilibrium Asset Prices over Infinite Horizons

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ABSTRACT. We obtain necessary and sufficient conditions for existence and uniqueness of equilibrium asset prices in discrete-time settings with infinite-horizon dividend streams. The results cover a range of traditional and modern asset pricing applications. Our findings are connected to the recent literature on stochastic discount factor decompositions via principal eigenpairs.

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1. INTRODUCTION

One of the most fundamental problems in economics is how to price an asset paying a stochastic cash flow. In frictionless discrete-time environments where arbitrage is absent, the equilibrium (ex-dividend) price process $\{P_t\}_{t \geqslant 0}$ associated with a dividend process $\{D_t\}_{t \geqslant 1}$ can be shown to obey

$$P_t = \mathbb{E}_t \, M_{t+1}(P_{t+1} + D_{t+1}) \quad \text{for all } t \geqslant 0, \tag{1}$$

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where $\{M_t\}_{t \geqslant 1}$ is the pricing kernel or stochastic discount factor (SDF) process of a representative investor.[1] The two questions addressed in this paper are:

1. When can we associate to each pair $\{D_t, M_t\}_{t \geqslant 1}$ a unique and finite equilibrium price process $\{P_t\}_{t \geqslant 0}$?
2. How can we characterize and evaluate these prices when they exist?

The first question can be formulated more precisely as follows: If we regard (1) as a map from the joint distribution of $\{D_t, M_t\}$ to a price process $\{P_t\}$ satisfying (1) almost surely, then when—and under what circumstances—is this mapping well defined? The second question concerns (a) existence and uniqueness of Markov equilibria and (b) connection to prices obtained by forward iteration.[2]

To provide motivation, consider the study of Epstein and Zin (1989), who write "....we have not demonstrated the consistency of our analysis with a general equilibrium framework such as Lucas' (1978) stochastic pure endowment economy. Such an extension would need to confront the questions of existence and uniqueness of equilibrium asset prices. Moreover, Lucas' contraction mapping techniques would not suffice for the same reasons that those techniques were inadequate in establishing Theorem 3.1. Thus we leave such an extension to a separate paper." Three decades later this issue remains unresolved. The same is true for a variety of asset pricing models, with a range of SDF and dividend specifications.

To understand the key ideas, think of the SDF M_{t+1} in (1), which serves to deflate payoffs in future states, as a random "contraction factor" around which one can build a contraction mapping argument, looking forwards in time. The operator in this contraction argument has as its fixed point an *equilibrium price function*, which is a map from the state process into an equilibrium price process. The operator itself is referred to below as the *equilibrium price operator*.

In the risk neutral case, where $M_{t+1} = \beta$ and $\beta < 1$, we have a uniform contraction rate of β in every state of the world, the equilibrium price operator is a contraction of modulus β, and and a proof of existence and uniqueness of the price process is

[1] See, for example, Kreps (1981), Cochrane and Hansen (1992), Hansen and Renault (2009) or Duffie (2010).

[2] Throughout the paper, we concern ourselves only with fundamental solutions rather than rational bubbles. For a recent discussion of the latter see Brunnermeier (2016).

trivial. Outside of this case, however, the SDF is random and $M_{t+1} > 1$ usually holds on a set of positive probability (since payoffs in bad states are highly valued). Contraction based arguments must confront this state of affairs.

An early example of a successful study in this context is Lucas (1978), who uses a change-of-variable argument to remove the stochastic component from the contraction coefficient in his model. The model in question has SDF

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}, \tag{2}$$

where $\{C_t\}$ is a stationary consumption process, β is a state independent discount component and u is a period utility function. By solving for $u'(C_t)P_t$ instead of P_t directly, Lucas (1978) obtains a modified pricing operator with contraction modulus equal to β.

This methodology can be generalized. For example, while Lucas (1978) assumes that dividends are stationary, one can make a similar argument in the case where dividend growth is stationary instead (see, e.g., Mehra and Prescott (2003)). Also, while Lucas (1978) requires that that utility and dividends are bounded, so that the equilibrium price operator acts in a a space of continuous bounded functions, similar results can be obtained in unbounded settings by truncating innovations or working with weighting functions and weighted supremum norms (see, e.g., Alvarez and Jermann (2005) or Brogueira and Schütze (2017)). Finally, the change-of-variable technique works not just for SDFs of the form (2), but also for any SDF that can be decomposed into the product of a state independent discount factor and a ratio of stationary factors.

Unfortunately, for many SDFs used in modern asset pricing applications, no such decomposition exists. Examples include those found in Epstein and Zin (1989), Bansal and Yaron (2004) and Schorfheide et al. (2017). Moreover, a substantial body of evidence shows that SDFs that possess a stationary factorization of the form discussed above cannot match asset price data in a number of dimensions (see, in particular, Borovička et al. (2016)).

When factoring M_{t+1} is not possible, one can still consider treating the entire SDF as a "random contraction factor." For example, even if $M_{t+1} > 1$ holds with positive probability, a contraction argument can still be constructed if, say, $\mathbb{E}_t M_{t+1} < 1$

in "most" states. This is the approach taken in Calin et al. (2005), who obtain existence and uniqueness results for equilibrium prices for a class of models involving habit formation by using contraction arguments in a space of integrable functions. While their ideas pertain to a very specific class of models, the approach of imposing conditions under which the equilibrium price operator acts as a contraction over a suitable class of functions can be developed without difficulty for many forms of asset pricing models.

However, just as the factorization based approaches are excessively restrictive, this alternative approach turns out to exclude many stable models with well defined equilibrium prices. Indeed, a fundamental problem for all of the approaches listed above is that the conditions required for contraction *in one step* are simply too tight. Modern asset pricing models aimed at quantitative applications often select parameterizations close to the boundary between stability and instability (see, e.g., sections 5.2–5.3). In such settings, conditions based on one step contractions fail and provide no useful information.

In this paper we adopt an alternative approach that considers instead discounting over the long run, using the implied n period state price deflator $\prod_{i=1}^{n} M_{t+i}$ with large n, and requiring only that contraction occurs "on average, eventually." For example, we show that, assuming a stationary dividend process and some basic regularity on the structure of the problem, the equilibrium price operator is a L_1 contraction at *some* finite power whenever

$$\lim_{n \to \infty} \left\{ \mathbb{E} \prod_{t=1}^{n} M_t \right\}^{1/n} < 1. \tag{3}$$

From this we obtain existence and uniqueness of equilibrium asset prices. (When dividend *growth* is stationary, rather than dividends, condition (3) is modified to feature a dividend-growth adjusted SDF.)

Results for elementary cases are easily recovered from condition (3). For example, in the risk neutral case $M_t = \beta$, the left hand side of (3) is just β. If $\{M_t\}$ is random but IID, then the limit in (3) is $\mathbb{E} M_t$ and hence $\mathbb{E} M_t < 1$ is sufficient for stability. Moreover, for transition independent SDFs such as (2), intermediate terms cancel when we take the product in (3), leading to simple conditions that recover (and extend) classical results.

For more complex SDFs, such as those arising from recursive preferences or habit formation, (3) can be evaluated either analytically, by calculating expectations and taking limits, or numerically, analogous to the way that spectral radius conditions of finite dimensional systems are examined in order to test stability. For example, we use this numerical approach to show equilibrium asset prices exist and are unique for the Epstein–Zin specifications used in Bansal and Yaron (2004) and Schorfheide et al. (2017).

The above discussion corresponds to one special case of our results, where the underlying function space is L_1. We also study outcomes in other function spaces, the benefit of which is that varying the function space introduces the possibility of imposing additional structure on the solution to the problem, such as continuity, or finiteness of second moments. Working in an abstract setting that includes such function spaces, we show that existence and uniqueness of equilibrium asset prices hold whenever $r(V) < 1$, where $r(V)$ is the spectral radius of the valuation operator V that maps future payoffs to current values via the set of state price deflators embedded in the stochastic discount factor. The L_1 results are a special case because, as we show using local spectral radius results, the limit in (3) is equal to $r(V)$ when the function space is L_1.

By using long run average contractions as determined by the spectral radius of the valuation operator, we obtain conditions that are very close to necessary—in contrast to the conditions based on one step contractions discussed above. For example, we show via an application of the Krein–Rutman theorem that if the valuation operator V is also compact, then $r(V) > 1$ implies that T^n is not a contraction for any n, and, more importantly, that no equilibrium price function exists. We also use local spectral radius conditions to weaken the compactness requirement in the case of L_1, since compactness is relatively stringent in this case.

Our focus on the spectral radius of the valuation operator also allows us to connect the problems studied here with the recent literature on stochastic discount factor decompositions analyzed in Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Hansen (2012); Borovička et al. (2016); Christensen (2017); Qin and Linetsky (2017) and other recent studies. These decompositions are used to extract a permanent growth component and a martingale component from the stochastic discount process, with the rate in the permanent growth component being driven by the principal eigenvalue of the operator associated with stochastic discount factor. While this literature uses the permanent growth component to gain insight on the structure of valuation for payoffs at alternative horizons, we connect the same permanent growth component to the existence and uniqueness of equilibrium asset prices in infinite horizon economies, and to the contractivity of the equilibrium asset pricing operator, as described above.

The rest of our paper is structured as follows. Section 2 sets up a general version of the problem. Section 3 states our main results. Applications are treated in sections 4 and 5. Proofs are deferred to the appendix.

2. Preliminaries

In this section we set out the existence and uniqueness problems for asset prices considered in the paper.

2.1. **Forward Looking Recursions.** Consider the forward looking model

$$Y_t = \mathbb{E}_t \left[\Phi_{t+1}(Y_{t+1} + G_{t+1}) \right] \quad \text{for all } t \geqslant 0, \tag{4}$$

where $\{\Phi_t\}$ and $\{G_t\}$ are given stochastic processes and $\{Y_t\}$ is endogenous. One version of (4) is the equilibrium price problem (1), where $\{G_t\}$ is cash flow and $\{\Phi_t\}$ is the SDF. Other versions arise when dividends are nonstationary, in which case dividing (1) by D_t leads to the natural endogenous variable being the price-dividend ratio $Y_t = P_t / D_t$, with $G_{t+1} = 1$ and $\Phi_{t+1} = M_{t+1} D_{t+1} / D_t$, or when alternative modifications to the fundamental asset pricing equation (1) are required.

We assume throughout that randomness in the economy is generated by an underlying *state process* $\{X_t\}$. In particular, we assume that $\{\Phi_t\}$ and $\{G_t\}$ admit the representations

$$\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1}) \quad \text{and} \quad G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1}) \tag{5}$$

where $\{\eta_t\}$ is a W-valued innovation sequence and ϕ and g are Borel measurable maps from $\mathsf{X} \times \mathsf{X} \times \mathsf{W}$ to $\mathbb{R}_+$. The sets X and W are arbitrary Polish spaces.[3] The state process is assumed to be stationary and Markovian. The representations in (5)

[3] The Polish assumption, which requires that X and W are separable and completely metrizable, is very weak and satisfied in all applications of which we are aware.

replicate the general multiplicative functional specifications considered in Hansen and Scheinkman (2009) and Hansen (2012), and are sufficient for all problems we consider.

A stochastic process $\{Y_t\}$ is called a *solution* of the pricing recursion (4) if it is nonnegative, finite $\mathbb{P}$-almost everywhere and (4) holds $\mathbb{P}$-almost surely.

Forward iteration yields the candidate solution

$$Y_t^F := \mathbb{E}_t \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} \Phi_{t+i} \, G_{t+n} \right], \tag{6}$$

which states that current price equals current expectation of total lifetime cash flow appropriately discounted. While (6) can be understood intuitively as the "fundamental solution," we refer to it for now as the *forward projection*, since it does not yet meet our definition of a solution (it could for example be infinite).

In what follows, the Borel sets of X are denoted by $\mathscr{B}$ and the stochastic kernel generating $\{X_t\}$ is denoted by Π.[4] The process $\{X_t\}$ is defined on some underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and satisfies $\mathbb{P}\{X_{t+1} \in B \mid X_t = x\} = \Pi(x, B)$ for all $x \in \mathsf{X}$ and $B \in \mathscr{B}$. The innovation process $\{\eta_t\}$ is assumed to be IID and independent of $\{X_t\}$. Each η_t has common distribution ν. The common distribution of each X_t is denoted by π. Relations such as (4) and (6) are understood as $\mathbb{P}$-almost sure equalities, while conditional expectations are with respect to the natural filtration generated by $\{X_t\}$.

2.2. **Markov Solutions.** Given the assumptions on the exogenous processes made in (5), it is natural to seek a Markov solution $Y_t^* = h^*(X_t)$ to the recursive pricing equation (4), where h^* is a fixed function in some candidate set $\mathcal{H}$. We require $\mathcal{H}$ to be a Banach lattice of real-valued functions on X with the usual notions of pointwise order, addition and scalar multiplication.[5] Let $\| \cdot \|$ denote the norm

[4] In particular, Π is a function from $(\mathsf{X}, \mathscr{B})$ to $[0, 1]$ such that $B \mapsto \Pi(x, B)$ is a probability measure on $(\mathsf{X}, \mathscr{B})$ for each $x \in \mathsf{X}$, and $x \mapsto \Pi(x, B)$ is $\mathscr{B}$-measurable for each $B \in \mathscr{B}$.

[5] A Banach lattice is a Riesz space that is also complete. To accommodate unbounded solutions, the set $\mathcal{H}$ will in some instances be identified with an L_p space. Elements of $\mathcal{H}$ are then equivalence classes of functions, rather than functions *per se* and pointwise statements such as equalities and inequalities are understood as almost everywhere requirements.

on $\mathcal{H}$ and let $\leqslant$ denote the partial order. Let $\mathcal{H}_+$ denote the *positive cone* of $\mathcal{H}$, consisting of all functions in $\mathcal{H}$ taking only nonnegative values.

As usual, the *norm* of a bounded linear operator L from $\mathcal{H}$ to itself is defined as

$$\|L\| := \sup_{\|f\|=1} \|Lf\|.$$

The *spectrum* $\sigma(L)$ of L is all scalars $\lambda \in \mathbb{C}$ such that $L - \lambda I$ fails to be bijective. A scalar λ is called an *eigenvalue* of L if there exists a nonzero $f \in \mathcal{H}$ such that $Lf = \lambda f$. The function f is then called an *eigenfunction*. The set $\sigma(L)$ is nonempty and compact in the complex plane, and every eigenvalue of L lies in $\sigma(L)$. The *spectral radius* of L is $r(L) := \max\{|\lambda| : \lambda \in \sigma(L)\}$. Since $\mathcal{H}$ is a Banach space, Gelfand's formula holds:

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}. \tag{7}$$

The operator L is called *compact* if the image under L of the unit ball in $\mathcal{H}$ lies in a compact subset of $\mathcal{H}$. L is called *positive* if it maps the positive cone $\mathcal{H}_+$ into itself.

Let $\mathcal{L}$ be all continuous linear functionals $\ell\colon \mathcal{H} \to \mathbb{R}$ such that $\ell(h) \geqslant 0$ whenever $h \in \mathcal{H}_+$. A function $h \in \mathcal{H}_+$ is called *quasi-interior* to $\mathcal{H}_+$ if $\ell(h) > 0$ for every nonzero $\ell \in \mathcal{L}$.

Example 2.1. If $\mathcal{H} = \mathscr{C}(\mathsf{X})$, the set of continuous bounded functions on X endowed with the supremum norm, then $\mathcal{H}$ is a Banach lattice with the property that any strictly positive function in $\mathscr{C}(\mathsf{X})$ is quasi-interior. The last statement can be checked using the Riesz representation theorem.

Example 2.2. Let π be the common marginal distribution of each X_t, as above, let p be a constant satisfying $p \geqslant 1$ and let $\mathcal{H} = L_p(\pi)$, the space of Borel measurable functions $g\colon \mathsf{X} \to \mathbb{R}$ such that

$$\|h\| := \left\{ \int |h(x)|^p \pi(\mathrm{d}x) \right\}^{1/p} \tag{8}$$

is finite. Functions equal π-almost everywhere are identified, so that (8) defines a norm on $L_p(\pi)$ and together with the pointwise order they form a Banach lattice. Every strictly positive function in $L_p(\pi)$ is quasi-interior to its positive cone, as follows from the Riesz representation theorem.

In the remainder of the paper, $\mathcal{H}$ will be one of the function spaces in examples 2.1–2.2. The L_p setting is more general than that of $\mathscr{C}(\mathsf{X})$, since every bounded measurable function on X lies in $L_p(\pi)$. Moreover, every $L_p(\pi)$ spaces lies in $L_1(\pi)$.[6]

Our findings on necessary conditions and principle eigenvalues of valuation operators rely in part on a result, due to Zabreiko et al. (1967) and Mirosława Zima (private communication), concerning the local spectral radii of positive compact linear operators acting on quasi-interior points:

Theorem 2.1 (Zabreiko–Krasnosel'skii–Stetsenko–Zima)**.** *Let h be an element of $\mathcal{H}_+$ and let $L\colon \mathcal{H} \to \mathcal{H}$ be a compact linear operator. If L is also positive and h is quasi-interior, then*

$$\lim_{n \to \infty} \|L^n h\|^{1/n} \to r(L). \tag{9}$$

The expression on the left hand side of (9) is called the *local spectral radius* of L at h. Theorem 2.1 gives conditions under which it equals the spectral radius. While a number of related results exist, theorem 2.1 is particularly useful because it allows us to consider spaces where the positive cone has empty interior (e.g., the L_p spaces in example 2.2). As we could find no complete proof published in English we include one in the appendix.

3. Results

In this section we set out the core theoretical results obtained in the paper.

3.1. The Equilibrium Price Operator.

Returning to the price recursion (4), any Markov solution $h^* \in \mathcal{H}$ must satisfy

$$h^*(x) = \int \int \phi(x, x', \eta) \left[h^*(x') + g(x, x', \eta)\right] \nu(\mathrm{d}\eta)\Pi(x, \mathrm{d}x') \tag{10}$$

for all $x \in \mathsf{X}$. In other words, h^* is a fixed point of the *equilibrium price operator* T defined by

$$Th = Vh + \hat{g}, \tag{11}$$

[6]While $L_1(\pi)$ is therefore the most general setting, the more specialized spaces give additional structure (for example, Markov solution in $L_2(\pi)$ have finite second moment, while solutions in $\mathscr{C}(\mathsf{X})$ are continuous). Hence we present most of our theory in the setting of a generic Banach lattice $\mathcal{H}$.

where

$$Vh(x) := \int h(x') \left[\int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}x') \qquad (x \in \mathsf{X}). \tag{12}$$

and

$$\hat{g}(x) := \int \int \phi(x, x', \eta) g(x, x', \eta) \nu(\mathrm{d}\eta) \Pi(x, \mathrm{d}x'). \tag{13}$$

We call V the *valuation operator* by analogy with the asset pricing models described above.

Assumption 3.1. Together, V, $\hat{g}$ and $\mathcal{H}$ have the following properties:

(a) The integral $\int |h| \, \mathrm{d}\pi$ is finite for all $h \in \mathcal{H}$.

(b) The valuation operator V maps $\mathcal{H}$ to itself and $\hat{g}$ is in $\mathcal{H}$.

Part (a) of assumption 3.1 means that any Markov solution $Y_t = h(X_t)$ with $h \in \mathcal{H}$ will have finite first moment. An assumption along these lines is almost unavoidable, since the problem definition embedded in the forward looking restriction (4) is stated in terms of conditional expectations, and conditional expectations are themselves defined in terms of unconditional expectations. Part (b) of assumption 3.1 implies that V is a nontrivial positive linear operator on $\mathcal{H}$ (nontrivial because are assuming an arbitrage free environment) and the equilibrium price operator T is a self-mapping both on $\mathcal{H}$ and $\mathcal{H}_+$. Any fixed point of T in $\mathcal{H}_+$ is called an *equilibrium price function*.

We can now state our first result concerning the equilibrium price operator:

Theorem 3.1. *Let assumption 3.1 hold. If the valuation operator V is compact, then the following statements are equivalent:*

(a) $r(V) < 1$

(b) There exists an $n \in \mathbb{N}$ such that T^n is a contraction on $(\mathcal{H}, \| \cdot \|)$.

(c) There exists a unique equilibrium price function h^* in $\mathcal{H}_+$ and $\lim_{n \to \infty} T^n h = h^*$ for every $h \in \mathcal{H}_+$.

That (a) implies (b) is relatively trivial, following directly from Gelfand's formula. Likewise, the fact that (b) implies (c) follows easily from a simple extension of the Banach contraction mapping theorem, which implies existence of a unique, globally attracting fixed point of T in $\mathcal{H}$. Since $\mathcal{H}$ is a Banach lattice, the set $\mathcal{H}_+$ is closed in $\mathcal{H}$ and, given that T is a self-mapping on $\mathcal{H}_+$, any fixed point must lie in $\mathcal{H}_+$. Compactness of V is not required either of these steps. On the other hand, the fact that (c) implies (a) requires compactness of the operator because it makes use of the Krein–Rutman theorem.

Theorem 3.1 leaves a number of open questions. For example, in the stable case $r(V) < 1$, what is the connection between the fixed point of T and the forward projection? In the unstable case $r(V) \geqslant 1$, does failure of contractivity imply failure of existence? Also, how can one evaluate the spectral radius conveniently? We turn to these issues below.

3.2. **Further Results for the Stable Case.** By theorem 3.1, the equilibrium price operator T has a unique, globally attracting fixed point h^* in $\mathcal{H}$ whenever $r(V) < 1$. The following theorem provides additional information. Note that compactness of V is not assumed.

Theorem 3.2. *If assumption 3.1 holds and $r(V) < 1$, then*

(a) The unique equilibrium price function h^* is equal to $\sum_{n \geqslant 0} V^n \hat{g}$.
(b) The process $\{Y_t^*\}$ defined by $Y_t^* := h^*(X_t)$ for all t solves (4).
(c) The forward projection Y_t^F in (6) is finite and equal to Y_t^* with probability one.
(d) If, in addition, $\lim_{n \to \infty} \int |h_n| \, \mathrm{d}\pi = 0$ whenever $\{h_n\} \subset \mathcal{H}$ and $\lim_{n \to \infty} \|h_n\| = 0$, then no other stationary Markov solution exists.

Part (a) of theorem 3.2 is an immediate consequence of the Neumann series lemma. Parts (b) and (c) use the Markov property of $\{X_t\}$ and assumption 3.1. The statement in part (d) that no other stationary Markov solution exists means that if $\{Y_t\}$ satisfies (4) and $\{Y_t\} = \{h(X_t)\}$ for some $h \in \mathcal{H}$, then$\{Y_t\}$ and $\{Y_t^*\}$ are *indistinguishable*. That is,

$$\mathbb{P}\{Y_t = Y_t^* \text{ for all } t\} = 1. \tag{14}$$

The condition on the norm of $\mathcal{H}$ in (d) says that convergence in this norm is stronger than convergence in $L_1(\pi)$. It is satisfied in the settings of examples 2.1–2.2. It can be further improved in some environments by dropping the Markov restriction. The appendix gives one example (see proposition 6.1).

3.3. **Further Results for the Unstable Case.** What happens when $r(V)$ exceeds unity? We know that T^n is not a contraction on $\mathcal{H}$ for any $n \in \mathbb{N}$ from theorem 3.1, and that the statement in (c) of theorem 3.1 fails. However, this implies neither absence nor multiplicity of fixed points. Moreover, it is not immediately obvious that the results in theorem 3.2 fail when $r(V) \geqslant 1$. To see why, consider the equilibrium price function $h^* = \sum_{n=0}^{\infty} V^n \hat{g}$ from (a) of theorem 3.2. Even if $r(V) > 1$, the operator V can still have some eigenvalues with modulus strictly less than unity. If $\hat{g}$ lies in a space spanned by the corresponding eigenfunctions, then the expression $\sum_{n=0}^{\infty} V^n \hat{g}$ can be well defined.

Nevertheless, $r(V) > 1$ does indeed imply divergence, as well as absence of a positive fixed point, when some regularity conditions are imposed. The next theorem gives details.

Theorem 3.3. *Let $\mathcal{H}$ be such that all strictly positive elements of $\mathcal{H}$ are quasi-interior to $\mathcal{H}_+$. If, in addition, V is compact and $r(V) > 1$, then no equilibrium price function exists in $\mathcal{H}_+$.*

The intuition behind this result is that $\hat{g}$ is positive by assumption and hence the dynamics of the iterates $V^n \hat{g}$ are similar to the dynamics of $V^n e$, where e is the principal eigenfunction (which is also positive by the Krein–Rutman theorem whenever the latter holds). If $r(V) > 1$, then $V^n e$ diverges. Hence $T^n e$ diverges. The connection between the dynamics of $V^n \hat{g}$ and dynamics of $V^n e$ are formalized using the local spectral radius result in theorem 2.1.

3.4. **Further Results for Integrable Functions.** In this section we specialize to the case $\mathcal{H} = L_1(\pi)$, where the latter is defined in example 2.2. This setting is important because the space $L_1(\pi)$ is large. For example, it allows us to tackle settings where the dividend process is unbounded, as is commonly assumed in applications. Moreover, as shown below, the local spectral radius condition yields a particularly simple expression for the spectral radius of the valuation operator V when we specialize to $L_1(\pi)$.

Note that $L_1(\pi)$ is not reflexive whenever the state space is infinite, and hence conditions for compactness of operators are stringent—and typically difficult to verify. This is potentially problematic when we wish to apply theorem 3.1 or theorem 3.3 because of the compactness requirement on V. In order to weaken this requirement, we introduce the following assumption:

Assumption 3.2. The state space X is endowed with some σ-finite Borel measure μ and the stochastic kernel Π for the state process has a density kernel $\pi(\cdot \mid \cdot)$ with respect to μ.[7] Moreover, the function $\psi \colon \mathsf{X} \to \mathbb{R}$ defined by

$$\psi(x) := \sup_{x' \in \mathsf{X}} \left\{ \int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \cdot \frac{\pi(x' \mid x)}{\pi(x')} \right\} \tag{15}$$

satisfies $\int \psi \, \mathrm{d}\pi < \infty$.

In a typical application, X will be a subset of $\mathbb{R}^d$ and μ is either Lebesgue measure or the counting measure. As discussed in the appendix, assumption 3.2 implies that the valuation operator V is continuous as a linear operator on $L_1(\pi)$. In fact, when (15) is valid, V is a Hille–Tamarkin operator on $L_1(\pi)$, which we show is sufficient to obtain an L_1 version of the local spectral radius result in theorem 2.1 without imposing compactness:

Proposition 3.4. *If the conditions of assumption 3.2 hold, then*

$$r^n_\Phi := \left\{ \mathbb{E} \prod_{i=1}^n \Phi_i \right\}^{1/n} \tag{16}$$

converges to $r(V)$ as $n \to \infty$.

The proof of proposition 3.4 begins with the expression $\lim_{n\to\infty} \|L^n h\|^{1/n}$ from (9) and then replaces L with V and $\|\cdot\|$ with the L_1 norm. It then uses the law of iterated expectations to simplify the resulting expression. This iterated expectation step works because, for positive elements of $L_1(\pi)$, the L_1 norm is additive.

Evidently, if $\lim_{n\to\infty} r^n_\Phi < 1$ and assumption 3.2 holds, then all the stability results in theorem 3.2 hold with $\mathcal{H} = L_1(\pi)$. This verifies one of the claims put forward in the introduction.

The next proposition shows that the compactness condition used to study the unstable case $r(V) > 1$ in theorem 3.3 can be weakened to assumption 3.2 when

[7] That is, π is a real-valued Borel measurable map on $\mathsf{X} \times \mathsf{X}$ satisfying $\Pi(x, B) = \int_B \pi(x' \mid x)\mu(\mathrm{d}x')$ for every $B \in \mathscr{B}$.

we are working in L_1. It also strengthens the nonexistence result in theorem 3.3 beyond Markov solutions. The proof uses of proposition 3.5 uses proposition 3.4.

Proposition 3.5. *If assumption 3.2 holds and $r(V) > 1$, then no equilibrium price function exists in L_1. Moreover, the recursive pricing equation (4) has no stationary solution with finite first moment.*

Remark 3.1. The result in (16) allows us to compute $r(V)$ via Monte Carlo. In particular, we can use the fact that

$$\lim_{m \to \infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} \Phi_i^{(j)} \right\}^{1/n} = r_\Phi^n \tag{17}$$

with probability one, where each $\Phi_1^{(j)}, \ldots, \Phi_n^{(j)}$ is an independently simulated path of $\{\Phi_t\}$. This follows from the strong law of large numbers combined with the fact that $Z_n \to Z$ $\mathbb{P}$-a.s. implies $g(Z_n) \to g(Z)$ $\mathbb{P}$-a.s. whenever $g \colon \mathbb{R} \to \mathbb{R}$ is continuous. Such a calculation avoids discretization and turns out to be accurate in the applications we consider (see, e.g., tables 1–2 on page 19).

4. Applications Part I: Stationary Dividends

We now turn to applications, focusing in this section on some well known models where dividends themselves are required to be stationary. (In section 5 we consider more empirically plausible assumptions.) The space of candidate solutions will be $L_1(\pi)$, where, as before, π is the marginal distribution of the state process $\{X_t\}$.

4.1. **Bounded Utility.**

Consider the asset pricing problem of Lucas (1978), where the price process obeys (1) and the stochastic discount factor is

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}. \tag{18}$$

Here C_t is consumption, β is a discount factor, and u is utility. Our first goal is to recover the existence and uniqueness result for prices obtained in Lucas (1978) using theorem 3.2.

To match the result in Lucas (1978), we take $Y_t := P_t\, u'(C_t))$ to be the endogenous object rather than P_t, where the latter represents the price of a claim to a stationary dividend stream $\{D_t\}$. Equation (1) yields

$$Y_t = \mathbb{E}_t \left[\beta (Y_{t+1} + u'(C_{t+1}) D_{t+1}) \right]. \tag{19}$$

In equilibrium, $C_t = D_t = d(X_t)$ for all t, where $\{X_t\}$ is the Markov state process and d is a given function. Equation (19) is a version of (4) with $\Phi_t = \beta$ and $G_{t+1} = u'(D_{t+1})D_{t+1}$. The functions ϕ and g in (5) are therefore

$$\phi(x, x', \eta) := \beta \quad \text{and} \quad g(x, x', \eta) := u'(d(x'))d(x'). \tag{20}$$

The function $\hat{g}$ defined in (13) becomes

$$\hat{g}(x) = \beta \int u'(d(x'))d(x')\Pi(x, \mathrm{d}x'). \tag{21}$$

Lucas (1978) assumes that u is concave and bounded, which in turn gives $0 \leqslant u'(d(x'))d(x') \leqslant N$ for some $N \in \mathbb{N}$. Hence $\hat{g}$ is bounded by βN, and therefore an element of $L_1(\pi)$. Moreover, $Vg = \beta g$ for any $g \in L_1(\pi)$, so V maps $L_1(\pi)$ to itself, implying that assumption 3.1 holds.

Moreover, the fact that $Vg = \beta g$ for any $g \in L_1(\pi)$ implies that the range space of V is one dimensional. In particular, the only eigenvalue of V is β, and hence the spectral radius $r(V)$ of V is also equal to β. (One can also obtain the same conclusion by observing that, since $\Phi_t = \beta$, the expression on the right hand side of (16) is equal to β for all n.) Theorem 3.2 then implies the existence of a unique stationary Markov equilibrium whenever $\beta < 1$. This is the same conclusion as proposition 3 of Lucas (1978).

4.2. **Constant Relative Risk Aversion.** The previous result relies on boundedness of utility, an assumption that is rarely satisfied in applications. We can drop this assumption provided that $\hat{g} \in L_1(\pi)$ continues to hold true. To give one example, consider the work of Brogueira and Schütze (2017), who use a weighted sup norm approach to extend the results of Lucas (1978) to the case $u(c) = c^{1-\gamma}/(1-\gamma)$ with $d(x) = \exp(x)$ and $\{X_t\}$ following

$$X_{t+1} = \rho X_t + b + \sigma \xi_{t+1}, \quad \{\xi_t\} \overset{\text{IID}}{\sim} N(0,1) \quad \text{and} \quad |\rho| < 1, \tag{22}$$

In this case the definitions of ϕ and g in (20) are unchanged, while $\hat{g}$ in (21) becomes

$$\hat{g}(x) = \beta \exp \left\{ (1-\gamma) \left(\rho x + b + \frac{(1-\gamma)\sigma^2}{2} \right) \right\}. \tag{23}$$

Since π is Gaussian, we have $\hat{g} \in L_1(\pi)$. The conditions of theorem 3.2 are again satisfied and hence a uniquely defined Markov solution $Y_t^* = h^*(X_t)$ exists. This recovers the main result of Brogueira and Schütze (2017) without their requirement of a positively correlated state process and several additional parameter restrictions. We can of course go further, dropping the AR(1) assumption and modifying the utility and dividend process specifications, provided that $\hat{g}(X_t)$ still has a finite first moment.

5. Applications Part II: Stationary Dividend Growth

The standard theory discussed in the previous section takes dividends to be stationary. Such models can be brought closer to the data by assuming instead that dividend growth is stationary. In this case we aim to solve for the price-dividend ratio $Q_t := P_t / D_t$, which, in view of (1), must satisfy

$$Q_t = \mathbb{E}_t \left[M_{t+1} \frac{D_{t+1}}{D_t} (Q_{t+1} + 1) \right]. \tag{24}$$

Let us summarize the implications of the preceding results for the solution of the price-dividend ratio Q_t in (24). Comparing (24) and (4), in this context we have

$$\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1}) = M_{t+1} \frac{D_{t+1}}{D_t} \tag{25}$$

and $G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1}) = 1$. Let

$$r_M := \lim_{n \to \infty} \left\{ \mathbb{E} \prod_{i=1}^{n} M_i \frac{D_n}{D_0} \right\}^{1/n}. \tag{26}$$

whenever the limit exists. The next result summarizes the L_1 theory of section 3.4 in terms of its implications for (24), the forward looking recursion for the price-dividend ratio.

Proposition 5.1. *If the conditions of assumption 3.2 hold, then the limit in (26) is well-defined and finite. Moreover,*

(a) $r_M = \lim_{n\to\infty} r_\Phi^n = r(V)$, *where* r_Φ^n *is as defined in (16) and* V *is the spectral radius of the valuation operator associated with (25).*

(b) If $r_M < 1$, *then a unique stationary Markov solution* $Q_t^* = h^*(X_t)$ *for (24) exists, where* $h^* \in L_1(\pi)$. *In particular, the conclusions of theorem 3.2 are valid.*

(c) *Conversely, if $r_M > 1$, then the price-dividend ratio equation (24) has no stationary solution with finite first moment.*

In the rest of this section we connect these results to several applications.

5.1. **CRRA Utility and Stochastic Dividend Growth.** Consider a benchmark asset pricing model as found in, say, Mehra and Prescott (2003), where $\ln D_{t+1} - \ln D_t = X_{t+1}$ for some stationary Markov process $\{X_t\}$. With $C_t = D_t$ and CRRA utility, this yields

$$\Phi_{t+1} := M_{t+1}\frac{D_{t+1}}{D_t} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} = \beta \exp\{(1-\gamma)X_{t+1}\}. \tag{27}$$

Hence $\phi(x, x', \eta) = \beta\exp((1-\gamma)x')$. Let $\{X_t\}$ follow the AR(1) process in (22).

Consider first assumption 3.2. Connecting the definition of ψ in that assumption to the present application, we have

$$\psi(x) \propto \sup_{x' \in \mathbb{R}} \exp\left\{(1-\gamma)x' - \frac{(x' - \rho x - b)^2}{2\sigma^2} + \frac{(x' - \mu_s)^2}{2\sigma_s^2}\right\}. \tag{28}$$

Here $\propto$ means "proportional to," μ_s is the stationary mean $b/(1-\rho)$ and σ_s^2 is the stationary variance $\sigma^2/(1-\rho^2)$. The stationary variance is larger than the conditional variance σ^2, so the supremum in (28) is finite. Simple arguments show that, after substituting the maximizing value of x' into the right hand side of (28), we have $\psi(x) \propto \exp(a_0 + a_1 x + a_2 x^2)$ for suitable constants a_i. As the stationary distribution of a Gaussian AR(1) process, π is itself Gaussian, and hence $\int \psi \, \mathrm{d}\pi$ is finite. In particular, assumption 3.2 and the conditions of proposition 5.1 hold.

As a consequence, existence of a finite price-dividend ratio depends on

$$r_M = \lim_{n\to\infty}\left\{\mathbb{E}\prod_{i=1}^{n}\Phi_i\right\}^{1/n} = \beta \lim_{n\to\infty} \exp\left\{(1-\gamma)\sum_{i=1}^{n} X_i\right\}^{1/n},$$

where the second equality is due to (27). Since $\{X_t\}$ obeys (22), we have

$$\exp\left\{(1-\gamma)\sum_{i=1}^{n} X_i\right\}^{1/n} = \exp\left\{(1-\gamma)\frac{\mu_n}{n} + \frac{(1-\gamma)^2 s_n^2}{2n}\right\},$$

where μ_n is the expectation of $\sum_{i=1}^{n} X_i$ and s_n^2 is its variance. Elementary manipulations yield

$$\frac{s_n^2}{n} = \frac{\sigma^2}{1-\rho^2}\left\{1 + \frac{2(n-1)}{n}\frac{\rho}{1-\rho} - \frac{2\rho^2}{n}\frac{1-\rho^{n-1}}{(1-\rho)^2}\right\}.$$

Hence

$$r_M = r(V) = \beta \exp\left\{(1-\gamma)\left[\frac{b}{1-\rho} + \frac{1-\gamma}{2}\frac{\sigma^2}{(1-\rho)^2}\right]\right\}. \tag{29}$$

Proposition 5.1 implies that a unique solution with finite first moment exists—and equals the forward projection—whenever (29) evaluates to strictly less than unity. If, on the other hand, $r_M > 1$, then no such solution exists.

The spectral radius $r(V)$ in (29) represents the discounted risk-adjusted growth rate of aggregate consumption C_t, and reveals the dual role parameter γ plays under CRRA utility. The term in brackets is the average risk-adjusted consumption growth rate. The rest of the expression constitutes intertemporal discounting, consisting of the time-preference parameter β and the contribution of intertemporal substitution captured by the term $1-\gamma$ multiplying the average growth rate.

Since an analytical expression for the spectral radius exists, the current setting provides a useful test case for the proposal to calculate the spectral radius of the valuation operator using Monte Carlo, via (17). Our interest is in examining whether or not the Monte Carlo based expressions are sufficiently accurate for moderate sample sizes. Tables 1–2 are supportive. The parameters here are chosen to match Mehra and Prescott (2003), with $\beta = 0.99$, $\rho = 0.941$, $\gamma = 2.5$, $\sigma = 0.000425$ and $b = 0.00104$ in table 1, while in table 2 we shifted γ to 2.0. The actual value of $r(V)$ indicated in the table caption is calculated from the closed form expression (29). The interpretation of n and m in the table is consistent with the left hand side of (17). In both tables the approximation is accurate up to five decimal places in all simulations.

5.2. **Long Run Risk.** Next we turn to an asset pricing model with Epstein–Zin utility and stochastic volatility in cash flow and consumption estimated by Bansal and Yaron (2004). Utility is given by

$$V_t = \left[(1-\beta)C_t^{1-1/\psi} + \beta\left\{\mathcal{R}_t\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}, \tag{30}$$

TABLE 1. Monte Carlo spectral radius estimates when $r(V) = 0.9659169$

	n = 400	n = 600	n = 800	n = 1000	n = 1200
m = 10000	0.9659158	0.9659146	0.9659107	0.9659181	0.9659126
m = 15000	0.9659137	0.9659146	0.9659154	0.9659193	0.9659161
m = 20000	0.9659119	0.9659144	0.9659159	0.9659166	0.9659142
m = 25000	0.9659167	0.9659127	0.9659155	0.9659161	0.9659127

TABLE 2. Monte Carlo spectral radius estimates when $r(V) = 0.9727279$

	n = 400	n = 600	n = 800	n = 1000	n = 1200
m = 10000	0.9727255	0.9727252	0.9727256	0.9727278	0.9727246
m = 15000	0.9727223	0.9727223	0.9727268	0.9727274	0.9727263
m = 20000	0.9727269	0.9727253	0.9727275	0.9727265	0.9727267
m = 25000	0.9727247	0.9727267	0.9727260	0.9727267	0.9727278

where $\{C_t\}$ is the consumption path extending on from time t and

$$\mathcal{R}_t(V_{t+1}) := (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}. \tag{31}$$

The parameter $\beta \in (0,1)$ is a time discount factor, while γ governs risk aversion and ψ is the elasticity of intertemporal substitution (EIS). Dividends and consumption grow according to

$$\begin{aligned}
\ln(C_{t+1}/C_t) &= \mu_c + z_t + \sigma_t\, \eta_{c,t+1}, \\
\ln(D_{t+1}/D_t) &= \mu_d + \alpha z_t + \phi_d\, \sigma_t\, \eta_{d,t+1}, \\
z_{t+1} &= \rho z_t + \phi_z\, \sigma_t\, \eta_{z,t+1}, \\
\sigma_{t+1}^2 &= \max\left\{ v\, \sigma_t^2 + d + \phi_\sigma\, \eta_{\sigma,t+1},\, 0 \right\}.
\end{aligned}$$

Here $\{\eta_{i,t}\}$ are IID and standard normal for $i \in \{d, c, z, \sigma\}$. The state X_t can be represented as $X_t = (z_t, \sigma_t)$. The stochastic discount factor associated with this model is

$$M_{t+1} = \beta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{W_{t+1}}{W_t - 1} \right)^{\theta - 1}$$

where W_t is the value of aggregate wealth and $\theta := (1-\gamma)/(1-1/\psi)$. See, for example, Bansal and Yaron (2004), p. 1503. Hence

$$\Phi_{t+1} := M_{t+1}\frac{D_{t+1}}{D_t} = \beta^\theta \frac{D_{t+1}}{D_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{W_{t+1}}{W_t - 1}\right)^{\theta-1}. \tag{32}$$

The wealth value process W_t can be represented as $W_t = w(X_t)$, where w solves the Euler equation

$$\beta^\theta\, \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\left(\frac{w(X_{t+1})}{w(X_t)-1}\right)^{\theta}\right] = 1.$$

Rearranging and using the expression for consumption growth given above, this equality can be expressed as

$$w(X_t) = 1 + \left\{\beta^\theta\, \mathbb{E}_t\left\{\exp\left[(1-\gamma)(\mu_c + z_t + \sigma_t\, \eta_{c,t+1})\right] w(X_{t+1})^\theta\right\}\right\}^{1/\theta}.$$

or

$$w(z,\sigma) = 1 + [Kw^\theta(z,\sigma)]^{1/\theta}, \tag{33}$$

where K is the operator

$$Kg(z,\sigma) = \beta^\theta \exp\left\{(1-\gamma)(\mu_c + z) + \frac{(1-\gamma)^2\sigma^2}{2}\right\}\Pi g(z,\sigma) \tag{34}$$

In this expression, $\Pi g(z,\sigma)$ is the expectation of $g(z_{t+1},\sigma_{t+1})$ given the state's law of motion, conditional on $(z_t,\sigma_t) = (z,\sigma)$.

The existence of a unique solution $w = w^*$ to (34) in the set of continuous bounded functions $\mathscr{C}(\mathsf{X})$ under the parameterization used in Bansal and Yaron (2004) is established in Borovička and Stachurski (2017) when the innovation terms $\{\eta_{i,t}\}$ are truncated (at arbitrarily large values), so that the state space is compact. In order to use this result, we study the same setting and seek an equilibrium price-dividend ratio function in $\mathscr{C}(\mathsf{X})$. In particular, we compute w^* using the iterative method described in Borovička and Stachurski (2017), recover W_t as $w^*(X_t)$ for each t and evaluate Φ_{t+1} via (32). The spectral radius $r(V)$ of the valuation operator is then obtained via (7).

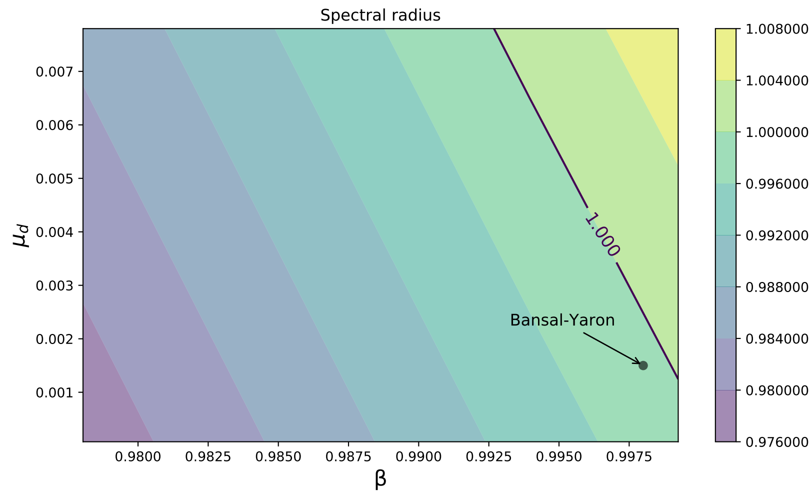

FIGURE 1. The spectral radius $r(V)$ in the Bansal–Yaron model

At the parameter values using in Bansal and Yaron (2004), we find that $r(V) = 0.9975$, implying the existence of a unique equilibrium price-dividend ratio function in $\mathscr{C}(\mathsf{X})$.[8] While this value is close to 1, significant shifts in parameters are required to cross the boundary $r(V) = 1$. For example, figure 1 shows the spectral radius $r(V)$ calculated at a range of parameter values in the neighborhood of the Bansal and Yaron (2004) specification via a contour map. The parameter β is varied on the horizontal axis, while μ_d is on the vertical axis. Other parameters are held fixed at the Bansal and Yaron (2004) values.

5.3. **Habit Persistence.** A number of papers including Abel (1990) and Campbell and Cochrane (1999) have studied models with consumption externalities, leading to SDFs of the form

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{s(H_t)}{s(H_0)}, \tag{35}$$

where s is a given function and H_t is the ratio of consumption to a social stock of past and present consumption. In Abel (1990), and in particular in the case

[8]Following Bansal and Yaron (2004), the parameters are $\gamma = 10.0$, $\beta = 0.998$, $\psi = 1.5$ $\mu_c = 0.0015$, $\rho = 0.979$, $\phi_z = 0.044$, $v = 0.987$, d =7.9092e-7, ϕ_σ =2.3e-6. $\mu_d = 0.0015$, $\alpha = 3.0$ and $\phi_d = 4.5$. See table IV on page 1489. The values of n and m in (17) in this calculation were set to 1,000 and 10,000 respectively. Although a range of alternative values were tested, none changed the main conclusion.

of "external" habit formation , the price dividend ratio implied by this stochastic discount factor satisfies the forward recursion (24) with

$$M_{t+1}\frac{D_{t+1}}{D_t} = k_0 \exp((1-\gamma)(\rho-\alpha)X_t) \tag{36}$$

where $k_0 := \beta \exp(b(1-\gamma) + \sigma^2(\gamma-1)^2/2)$ and α is a preference parameter. The connection between (35) and (36) is detailed in section 2.1 of Calin et al. (2005). The state sequence $\{X_t\}$ obeys (22) with $b := x_0 + \sigma^2(1-\gamma)$. Here x_0 is a parameter indicating mean constant growth rate of the dividend of the asset. See Calin et al. (2005) for details. In our notation,

$$\phi(x, x', \eta) = k_0 \exp((1-\gamma)(\rho-\alpha)x). \tag{37}$$

We wish to exploit the results in proposition 5.1, which necessitates checking condition (15). We have

$$\psi(x) \propto \sup_{x' \in \mathbb{R}} \exp\left\{(1-\gamma)(\rho-\alpha)x + \frac{-(x'-\rho x - b)^2}{2\sigma^2} + \frac{(x'-\mu_s)^2}{2\sigma_s^2}\right\}$$

Similar analysis to that conducted in section 5.1 shows that ψ is in $L_1(\pi)$, and hence the conditions of proposition 5.1 hold.

The spectral radius can also be computed in similar fashion to section 5.1, yielding

$$r(V) = k_0 \exp\left((1-\gamma)(\rho-\alpha)\frac{b}{1-\rho} + \frac{(1-\gamma)^2(\rho-\alpha)^2}{2}\frac{\sigma^2}{(1-\rho)^2}\right).$$

By proposition 5.1, a unique solution with finite first moment exists whenever $r(V) < 1$ and fails to exist when $r(V) > 1$.

To give some basis for comparison, let us contrast the condition $r(V) < 1$ with the sufficient condition for existence and uniqueness of an equilibrium price-dividend ratio found in proposition 1 of Calin et al. (2005), which implies a one step contraction. Since the spectral radius condition requires only eventual contraction and is almost necessary we can expect it to be weaker than the condition of Calin et al. (2005).

Figure 2 supports this conjecture. Each sub-figure shows the results of either the one step or the spectral radius test at a range of parameter values. The left sub-figure shows the one step test values obtained by evaluating the expression in equation (7) of Calin et al. (2005). The right sub-figure gives the spectral radius

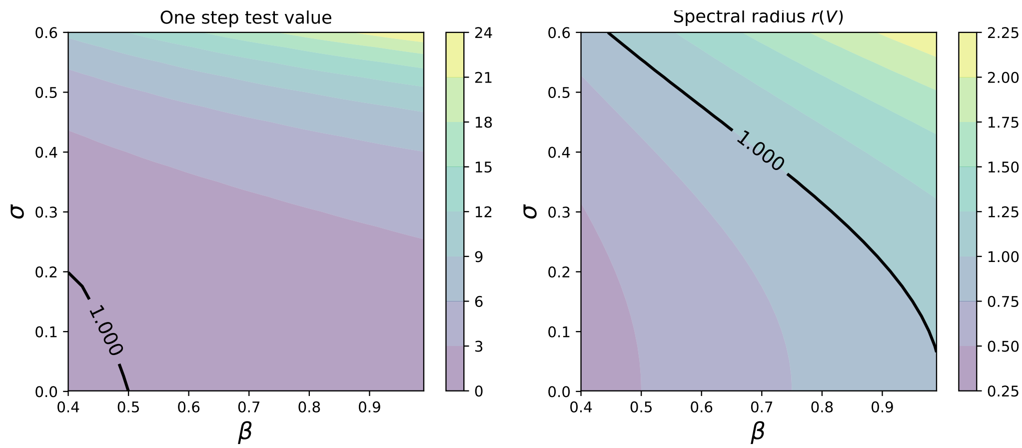

FIGURE 2. One step test value and spectral radius $r(V)$ with $x_0 = 0.05$

$r(V)$. The horizontal and vertical axes show grid points for the parameters β and σ respectively. Pairs (β, σ) with test values strictly less than one (points to the south west of the 1.0 contour line) are where the respective condition holds. Points to the north west of this contour line are where it fails. Inspection of the figure shows that the sufficient condition in Calin et al. (2005) requires an unrealistic discount factor, and fails for many parameterizations that do in fact have unique stationary Markov equilibria.[9]

6. APPENDIX

Remaining proofs are completed below.

Proof of theorem 3.1. To see that (a) implies (b), suppose that $r(V) < 1$. Using Gelfand's formula, choose $n \in \mathbb{N}$ such that $\|V^n\| < 1$. Then, for any $h, h' \in \mathcal{H}$ we have

$$\|T^n h - T^n h'\| = \|V^n h - V^n h'\| = \|V^n (h - h')\| \leqslant \|V^n\| \cdot \|h - h'\|.$$

To go from (b) to (c), observe that $\mathcal{H}_+$ is closed in $\mathcal{H}$, since $\mathcal{H}$ is a Banach lattice. Since V is positive, it maps $\mathcal{H}_+$ to itself. The remaining results follow from a well-known extension to the Banach contraction mapping theorem (see, e.g., p. 272 of Wagner (1982)).

[9]The parameters held fixed in figure 2 are, following Calin et al. (2005), $\rho = 0.98$, $\gamma = 2.5$, $x_0 = 0.05$ and $\alpha = 1$.

To show (c) implies (a), we will make use of the Krein–Rutman theorem applied to the operator V. In doing so, we note that the positive cone of $\mathcal{H}$ is reproducing in $\mathcal{H}$, since every $h \in \mathcal{H}$ can be expressed as the difference between $\max\{h, 0\}$ and $-\min\{h, 0\}$. These functions lie in $\mathcal{H}$ by the Banach lattice assumption and are obviously nonnegative.

Now suppose that (a) fails, so $r(V) \geqslant 1$. Since V is both positive and compact, the Krein–Rutman theorem (see theorem 41.2 of Zaanen (1997)) implies existence of an eigenfunction e such that $Ve = r(V)e$. Hence

$$\|T^n 0 - T^n e\| = \|V^n 0 - V^n e\| = \|0 - r(V)^n e\| = r(V)^n \|0 - e\|$$

As e is an eigenfunction it must be nonzero, so we have two points in $\mathcal{H}_+$ such that the distance between them fails to converge to zero. Such sequences cannot converge to the same point, so (c) cannot hold. □

Proof of theorem 3.2. Let I denote the identity map on $\mathcal{H}$. Since $r(V) < 1$, there exists an $i \in \mathbb{N}$ such that $\|V^i\| < 1$. As $\mathcal{H}$ is a Banach space, the Neumann series theorem then implies that $(I - V)^{-1}$ is well-defined on $\mathcal{H}$ and equals $\sum_{i=0}^{\infty} V^i$ (see, e.g., theorem 2.3.1 and corollary 2.3.3 of Atkinson and Han (2009)). In particular, $h^* = \sum_{n=0}^{\infty} V^n \hat{g}$ is a well-defined element of $\mathcal{H}$ (using assumption 3.1, which gives $\hat{g} \in \mathcal{H}$). Moreover, $h^* = (I - V)^{-1} \hat{g}$ and hence

$$h^* = Vh^* + \hat{g}. \tag{38}$$

Part (a) is now established.

Regarding (b), let $\{Y_t^*\}$ be defined by $Y_t^* = h^*(X_t)$ for all t. To show that this process solves (4), we need to show that it is nonnegative, almost everywhere finite and satisfies (4) with probability one. The first two claims follow immediately from $Y_t^* = \int h^*(X_t)$ and our assumptions on $\mathcal{H}_+$. Regarding the third, observe that, for any fixed $t \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathbb{E}_t \left[\Phi_{t+1}(Y_{t+1}^* + G_{t+1})\right] &= \mathbb{E}_t \left\{\phi(X_t, X_{t+1}, \eta_{t+1})[Y_{t+1}^* + g(X_t, X_{t+1}, \eta_{t+1})]\right\} \\
&= \int \int \phi(X_t, x', \eta) h^*(x') \nu(d\eta) \Pi(X_t, \mathrm{d}x') + \hat{g}(X_t) \\
&= Vh^*(X_t) + \hat{g}(X_t)
\end{aligned}$$

In view of (38), this last expression evaluates to $h^*(X_t) = Y_t^*$. Thus, $\{Y_t^*\}$ satisfies (4), and claim (b) is established.

Regarding (c), as a first step we show that

$$V^{n-1}\hat{g}(X_t) = \mathbb{E}_t \prod_{i=1}^{n} \Phi_{t+i} G_{t+n} \tag{39}$$

with probability one for all $n \in \mathbb{N}$. To see this, consider first the case $n = 1$. By the definition of $\hat{g}$ we have

$$\begin{aligned} V^0\hat{g}(X_t) = \hat{g}(X_t) &= \int\int \phi(X_t, x', \eta) g(X_t, x', \eta) \nu(\mathrm{d}\eta) \Pi(X_t, \mathrm{d}x') \\ &= \mathbb{E}_t\, \phi(X_t, X_{t+1}, \eta_{t+1}) g(X_t, X_{t+1}, \eta_{t+1}) = \mathbb{E}_t\, \Phi_{t+1} G_{t+1}. \end{aligned}$$

Thus, (39) holds when $n = 1$. Now suppose it holds at arbitrary $n \in \mathbb{N}$. We claim it also holds at $n + 1$. Indeed,

$$\begin{aligned} V^n\hat{g}(X_t) &= \int\int \phi(X_t, x', \eta) V^{n-1}\hat{g}(x') \nu(\mathrm{d}\eta) \Pi(X_t, \mathrm{d}x') \\ &= \mathbb{E}_t\, \phi(X_t, X_{t+1}, \eta_{t+1}) V^{n-1}\hat{g}(X_{t+1}) \end{aligned}$$

Using the induction hypothesis and the law of iterated expectations,

$$V^n\hat{g}(X_t) = \mathbb{E}_t\, \Phi_{t+1} \mathbb{E}_{t+1} \prod_{i=2}^{n} \Phi_{t+i} G_{t+n} = \mathbb{E}_t \prod_{i=1}^{n} \Phi_{t+i} G_{t+n}.$$

Thus, (39) holds for all n.

To complete the proof of (c), we use $h^* = \sum_{n \geqslant 0} V^n \hat{g}$ and (39) to obtain

$$h^*(X_t) = \sum_{n=1}^{\infty} V^{n-1}\hat{g}(X_t) = \sum_{n=1}^{\infty} \mathbb{E}_t \prod_{i=1}^{n} \Phi_{t+i} G_{t+n} = \mathbb{E}_t \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} \Phi_{t+i} G_{t+n} \right].$$

The last equality in the previous display follows from the monotone convergence theorem.

Thus, $h^*(X_t)$ is indeed equal almost surely to Y_t^F in (6). The forward projection is finite almost surely because of this equality and the finite first moment of $h^*(X_t)$, which was proved in part (b).

Regarding part (d) of theorem 3.2, let $\{Y_t\}$ be a sequence satisfying both (4) and $Y_t = h(X_t)$ for some $h \in \mathcal{H}$. Forward iteration then gives

$$Y_t - Y_t^* = \mathbb{E}_t \left[\prod_{i=1}^{n} \Phi_{t+i} [h(X_{t+n}) - h^*(X_{t+n})] \right].$$

This is equivalent to

$$Y_t - Y_t^* = V^n q(X_t) \quad \text{where} \quad q := h - h^*.$$

Thus, for any $n \in \mathbb{N}$ we have

$$\mathbb{E}|Y_t - Y_t^*| = \mathbb{E}|V^n q(X_t)|$$

By the spectral radius condition $r(V) < 1$ we have $\|V^n q\| \leqslant \|V^n\| \|q\| \to 0$ as $n \to \infty$. By the properties on the norm $\|\cdot\|$ imposed in the statement of claim (d), this yields $\mathbb{E}|V^n q(X_t)| \to 0$ as $n \to \infty$, from which we conclude that $\mathbb{E}|Y_t - Y_t^*| = 0$, and hence $\mathbb{P}\{Y_t = Y_t^*\} = 1$. Since the intersection of countable many probability one sets has probability one, the statement in (14) is established. Claim (d) is thus verified and the proof of theorem 3.2 is complete. □

Proposition 6.1. *If $\mathcal{H} = L_2(\pi)$ and the conditions of theorem 3.2 hold, then any stationary solution to (4) with finite second moment is indistinguishable from $\{Y_t^*\}$.*

Proof of proposition 6.1. Let $\{Y_t\}$ be a stationary solution to (4) with finite second moment. Fixing t and iterating on (4) yields

$$Y_t = \mathbb{E}_t \left[\sum_{j=1}^n \prod_{i=1}^j \Phi_{t+i} G_{t+j} + \prod_{i=1}^n \Phi_{t+i} Y_{t+n} \right] \quad \text{for any } n \in \mathbb{N}.$$

Subtracting the analogous expression for Y_t^* gives

$$\mathbb{E}|Y_t - Y_t^*| \leqslant \mathbb{E} \left[\prod_{i=1}^n \Phi_{t+i} \cdot |Y_{t+n} - Y_{t+n}^*| \right].$$

From this bound and the Cauchy–Schwarz inequality we have

$$\mathbb{E}|Y_t - Y_t^*| \leqslant \sqrt{\mathbb{E} \left[\prod_{i=1}^n \Phi_{t+i}^2 \right] \mathbb{E}(Y_{t+n} - Y_{t+n}^*)^2}.$$

By assumption, both $\{Y_t\}$ and $\{Y_t^*\}$ are stationary and have finite second moments. Moreover, by the definition of V we have $\mathbb{E} \prod_{i=1}^n \Phi_{t+i}^2 = \|V^n \mathbb{1}\|^2$ where $\mathbb{1} \in L_2(\pi)$ is unity everywhere on X and $\|\cdot\|$ is the $L_2(\pi)$ norm. Since $\|V^n \mathbb{1}\| = \|V^n\| \to 0$ as $n \to \infty$ by the spectral radius assumption, we conclude that $\mathbb{E}|Y_t - Y_t^*| = 0$. Hence $Y_t = Y_t^*$ with probability one. Since the time index is countable, it follows that $\{Y_t\}$ and $\{Y_t^*\}$ are indistinguishable, as was to be shown. □

In the following result we use the fact that the space $(\mathcal{H}, \|\cdot\|)$ is assumed to be a Banach lattice when endowed with the pointwise order, which implies that the positive cone (the functions in $\mathcal{H}$ taking nonnegative values) is both normal and reproducing.[10]

Proof of theorem 2.1. Let h and L be as in the statement of the theorem and let $\mathcal{H}_+$ be the positive cone of $\mathcal{H}$. Set

$$r(h, L) := \limsup_{n \to \infty} \|L^n h\|^{1/n}.$$

From the definition of $r(L)$ it suffices to show that $r(h, L) \geqslant r(L)$. To this end, let λ be a constant satisfying $\lambda > r(h, L)$ and let

$$h_\lambda := \sum_{n=0}^{\infty} \frac{L^n h}{\lambda^{n+1}}. \tag{40}$$

The point h_λ is a well-defined element of $\mathcal{H}_+$, as follows from

$$\limsup_{n \to \infty} \|L^n h\|^{1/n} < \lambda$$

and Cauchy's root test for convergence. It is also quasi-interior, since the sum in (40) includes the quasi-interior element h, and since L maps $\mathcal{H}_+$ into itself. Moreover, by the standard Neumann series theory of linear equations (e.g., Krasnosel'skii et al. (2012), theorem 5.1), the point h_λ also has the representation $h_\lambda = (\lambda I - L)^{-1} h$, from which we obtain $\lambda h_\lambda - L h_\lambda = h$. Because $h \in \mathcal{H}_+$, this implies that

$$L h_\lambda \leqslant \lambda h_\lambda. \tag{41}$$

Applying inequality (41), compactness of L, quasi-interiority of h_λ and theorem 5.5 (a) of Krasnosel'skii et al. (2012), we must have $r(L) \leqslant \lambda$. Since this inequality was established for an arbitrary λ satisfying $\lambda > r(h, L)$, we conclude that $r(h, L) \geqslant r(L)$. $\square$

Now we can return to the

[10] The positive cone of a partially ordered normed linear space is called *reproducing* if its linear span equals the whole space. It is called *normal* if there exists a finite constant N such that $\|g\| \leqslant N\|h\|$ whenever $g \leqslant h$.

Proof of theorem 3.3. Let $\mathcal{H}$ and V have the stated properties and suppose that $h \in \mathcal{H}_+$ and h solves the functional equation (10), which is to say that $h = Vh + \hat{g}$. Iterating on this equation, we have

$$h = \hat{g} + V\hat{g} + \cdots + V^n\hat{g} + V^{n+1}h.$$

Since $\mathcal{H}$ is a Banach lattice and all terms on the right hand side of this expression are nonnegative, we must have $\|V^n\hat{g}\| \leqslant \|h\|$ for all $n \in \mathbb{N}$.

On the other hand, $\hat{g}$ is a strictly positive element of $\mathcal{H}_+$ and therefore quasi-interior to $\mathcal{H}_+$. Applying theorem 2.1, we have $\|V^n\hat{g}\|^{1/n} \to r(V)$ as $n \to \infty$. Since $r(V) > 1$, this implies that $\|V^n\hat{g}\| \to \infty$. Contradiction. □

Proposition 6.2. *If assumption 3.2 holds and then the valuation operator V is a bounded linear operator on $L_1(\pi)$ and, for every strictly positive function $h \in L_1(\pi)$, we have*

$$\lim_{n\to\infty} \left\{ \int V^n h \, \mathrm{d}\pi \right\}^{1/n} = r(V). \tag{42}$$

Proof of proposition 6.2. Recall that an operator $T\colon L_1(\pi) \to L_1(\pi)$ is called a Hille–Tamarkin operator if T takes the form

$$Th(x) = \int k(x, x')h(x')\pi(\mathrm{d}x')$$

for some jointly measurable kernel k on $\mathsf{X} \times \mathsf{X}$ and, in addition, k satisfies the finite double norm property

$$\int \sup_{x' \in \mathsf{X}} |k(x, x')| \pi(\mathrm{d}x) < \infty. \tag{43}$$

Hille–Tamarkin operators on $L_1(\pi)$ have the property that T^2 is compact whenever π is σ-finite, as it is in our case. See, for example, theorem 4.5 of Grobler (1970).

Under the conditions of proposition 6.2, the valuation operator V is a Hille–Tamarkin operator. Indeed, V can be expressed as

$$\begin{aligned} Vh(x) &= \int h(x') \int \phi(x, x', \eta)\nu(\mathrm{d}\eta)\pi(x' \,|\, x) \, \mathrm{d}x' \\ &= \int h(x') \int \phi(x, x', \eta)\nu(\mathrm{d}\eta) \frac{\pi(x' \,|\, x)}{\pi(x')} \pi(x') \, \mathrm{d}x'. \end{aligned}$$

With

$$k(x, x') = \int \phi(x, x', \eta)\nu(\mathrm{d}\eta) \frac{\pi(x' \,|\, x)}{\pi(x')} \pi(x')$$

and the conditions of proposition 6.2 in force, the integrability condition (43) is satisfied, and V is Hille–Tamarkin as claimed.

As a result, V^2 is a compact linear operator on $L_1(\pi)$. Evidently it is positive. Since h is assumed to be everywhere positive and hence is quasi-interior, it follows from theorem 2.1 that $\{\int V^{2n} h \, \mathrm{d}\pi\}^{1/n}$ converges to $r(V^2)$. But $r(V^2) = r(V)^2$, so

$$\left\{ \int V^{2n} h \, \mathrm{d}\pi \right\}^{1/(2n)} \to r(V).$$

By our assumptions on V we know that Vh inherits the quasi-interiority of h, so another application of theorem 2.1, this time to V^2 with initial condition Vh, yields

$$\left\{ \int V^{2n} Vh \, \mathrm{d}\pi \right\}^{1/n} = \left\{ \int V^{2(n+1)} h \, \mathrm{d}\pi \right\}^{1/n} \to r(V)^2.$$

$$\therefore \quad \left\{ \int V^{2(n+1)} h \, \mathrm{d}\pi \right\}^{1/(2n)} \to r(V).$$

Some straightforward analysis then shows that

$$\left\{ \int V^{2(n+1)} h \, \mathrm{d}\pi \right\}^{1/(2(n+1))} \to r(V)$$

is also valid. We have now shown that $\{\int V^k h \, \mathrm{d}\pi\}^{1/k}$ converges to $r(V)$ along both even and odd subsequences. Hence the sequence itself converges to $r(V)$, and (42) is confirmed.

The second claim in proposition 6.2 is evident from the validity of (42) and the proof of theorem 3.3. □

Proof of proposition 3.4. Let $\mathbb{1}$ be equal to unity everywhere on X. Simple manipulations show that $V^n \mathbb{1}(X_t) = \mathbb{E}_t \prod_{i=1}^n \Phi_{t+i}$. By this equality and the law of iterated expectations,

$$r_\Phi = \left\{ \mathbb{E} \prod_{i=1}^n \Phi_{t+i} \right\}^{1/n} = \left\{ \mathbb{E}\, \mathbb{E}_t \prod_{i=1}^n \Phi_{t+i} \right\}^{1/n} = \left\{ \int V^n \mathbb{1} \, \mathrm{d}\pi \right\}^{1/n} \to r(V),$$

as $n \to \infty$, with the convergence due to positivity of $\mathbb{1}$ and (42). □

Proof of proposition 3.5. Let the conditions of the proposition hold but suppose, contrary to the claim in the proposition, that $\{Y_t\}$ is a stationary solution to the price recursion (4) with finite first moment. Iterating on (4) gives

$$Y_t = \mathbb{E}_t \left[\sum_{n=1}^{m} \prod_{i=1}^{n} \Phi_{t+i} G_{t+n} + \prod_{i=1}^{m} \Phi_{t+i} Y_{t+m} \right] \quad \text{for any } m \in \mathbb{N}.$$

Taking expectations and using the law of iterated and the nonnegativity of $\{Y_t\}$, we have

$$\mathbb{E} Y_t \geqslant \left[\sum_{n=1}^{m} \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} G_{t+n} \right] \quad \text{for any } m \in \mathbb{N}.$$

Together, (16) in proposition 3.4 and the converse component of the Cauchy root criterion imply that this sum diverges. Hence $\mathbb{E} Y_t = \infty$, contradicting our assumption that the solution has finite first moment. □

Proof of proposition 5.1. Let the conditions of proposition 6.2 hold. Regarding part (a) of proposition 5.1, that $r_M = r_\Phi$ in the present setting follows immediately from $\Phi_{t+1} = M_{t+1} D_{t+1} / D_t$. That $r_\Phi = r(V)$ was shown in proposition 3.4.

Regarding part (b), from part (a) we have $r_M < 1$ implies $r(V) < 1$. Hence we can employ theorem 3.2 and conclude the claim in (b) is true provided that assumption 3.1 is satisfied. That assumption 3.1 is true when $\mathcal{H} = L_1(\pi)$ follows from proposition 6.2, which ensures us that V is a bounded linear operator on $L_1(\pi)$.[11]

That part (c) holds follows directly from $r_M = r(V)$ and proposition 3.5. □

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[11] An immediate consequence is that V maps $L_1(\pi)$ to itself. Moreover, $\hat{g} \in L_1(\pi)$ because $\hat{g} = V\mathbb{1}$ and $\mathbb{1} \in L_1(\pi)$ since π is a probability measure.

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