Economic Dynamics: Theory and Computation

Second Edition

Additional Exercises

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(Solutions available on request)

PRELIMINARIES

The questions below use the material in the textbook plus the following two additional results.

Theorem 1 (Neumann Series Lemma). *Let A be $n \times n$, let b be $n \times 1$ and let I be the $n \times n$ identity matrix. If $r(A) < 1$, then $I - A$ is nonsingular and $x = Ax + b$ has the unique solution*

$$x^* = (I - A)^{-1} b = \sum_{k \geq 0} A^k b.$$

In the theorem above, $r(A)$ is the spectral radius of A, defined by

$$r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} \tag{1}$$

Here $|\lambda|$ indicates the modulus of the complex number λ.

In the next result, we take S and T to be self-maps on $M \subset \mathbb{R}^n$ and let $\leq$ be the pointwise partial order on $\mathbb{R}^n$. We say that T dominates S on M if $Sx \leq Tx$ for all $x \in M$.

Theorem 2. *If T dominates S on M and, in addition, T is monotone increasing and globally stable on M, then its unique fixed point dominates any fixed point of S.*

Also, for a given pair of distributions ϕ, ψ on $\mathbb{R}$, we say that ψ *first order stochastically dominates* ϕ if

$$\int u(x)\phi(\mathrm{d}x) \leq \int u(x)\psi(\mathrm{d}x) \;\text{ for every increasing bounded function } u \colon \mathbb{R} \to \mathbb{R}$$

Finally, we say that ψ is a *mean-preserving spread* of ϕ if there exists a pair of random variables (Y, Z) such that

$$\mathbb{E}\,[Z \,|\, Y] = 0, \quad Y \overset{\mathrm{d}}{=} \phi \quad \text{and } \; Y + Z \overset{\mathrm{d}}{=} \psi$$

In other words ψ is a mean-preserving spread of ϕ if it adds noise without changing the mean.

Question 1. Let $D = [0, 1]$ and let $f : D \to \mathbb{R}$ be defined by $f(x) = 1 - |x - 0.5|$. List the maximizers and minimizers of f on D, if any. Explain your reasoning.

Solution 1. Since $-|x - 0.5|$ is always negative, the unique maximizer is the x that makes this zero, which is 0.5. The minimizers are the two values that make this $|x - 0.5|$ as large as possible, which are 0 and 1.

Question 2. Give an example of a function f from $\mathbb{R}$ to $\mathbb{R}$ that has neither a maximizer nor a minimizer. Explain why the function has this property.

Solution 2. An example function is $f(x) = x$. No x can be a maximizer because $x + 1$ has a greater value. No x can be a minimizer because $x - 1$ has a smaller value.

Question 3. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$. Suppose that a and b are both maximizers of f on D. Is it necessarily true that $f(a) = f(b)$? Why or why not?

Solution 3. This is true. To see why, suppose that a and b are both maximizers of f on D. Since a is a maximizer it satisfies $f(a) \geq f(x)$ for all $x \in D$, and, in particular, $f(a) \geq f(b)$. A similar argument gives $f(b) \geq f(a)$. Hence $f(b) = f(a)$.

Question 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|$. Is f a bijection? Why or why not?

Solution 4. This is not a bijection. For example, it is not onto. Indeed, since $f(x) \geq 0$ for all x, there is no x that maps to -1, say.

Question 5. An infinitely lived worker can either be unemployed (state 0) or employed (state 1). If the worker is currently unemployed, then she remains unemployed with probability α and becomes employed with probability $1 - \alpha$. If the worker is currently employed, then she remains employed with probability β and becomes unemployed with probability $1 - \beta$. Let $X_t \in \{0, 1\}$ denote the current state of the worker.

1. Write down the stochastic kernel p for the Markov chain $(X_t)_{t \geq 0}$ as a matrix.
2. Does p have a stationary distribution? Why or why not?
3. Give a set of conditions on α, β such that p has exactly one stationary distribution. Connect your condition to the discussion of Markov chains in the lecture slides or course notes.

Solution 5. Possible solutions are as follows:

1. The stochastic kernel is
$$p = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \beta & \beta \end{pmatrix}$$
2. Yes, every finite state Markov chain has at least one stationary distribution, as stated in the text.
3. If $0 < \alpha, \beta < 1$, then a unique stationary distribution exists. This is because, under the stated condition, p is everywhere positive. Positivity is a sufficient condition for global stability, which implies uniqueness.

Question 6. Consider a worker whose log income obeys

$$y_{t+1} = \rho y_t + b + c\xi_{t+1} \tag{2}$$

where y_0 is a constant and $(\xi_t)_{t \geq 1}$ is an IID stochastic process with $\mathbb{E}\, \xi_t = 0$ and $\mathbb{E}\, \xi_t^2 = 1$. Let $\mu_t := \mathbb{E}\, y_t$ and let σ_t^2 be the variance of y_t.

1. Provide a recursive relationship linking μ_{t+1} and μ_t.

2. State a condition under which $\{\mu_t\}$ converges and provide an expression for the limit.

3. Provide a recursive relationship linking σ_{t+1}^2 and σ_t^2.

4. State a condition under which $\{\sigma_t^2\}$ converges and provide an expression for the limit.

Solution 6. Possible solutions are as follows:

1. The link is $\mu_{t+1} = \rho \mu_t + b$.

2. This sequence converges to $\bar{\mu} := b/(1-\rho)$ whenever $|\rho| < 1$.

3. The link is $\sigma_{t+1}^2 = \rho^2 \sigma_t^2 + c^2$.

4. This sequence converges to $\bar{\sigma}^2 := c^2/(1-\rho^2)$ whenever $|\rho| < 1$.

Question 7. Continuing on from question 6, suppose in addition that $c > 0$ and each ξ_t has density ϕ on $\mathbb{R}$.

1. Letting ψ_t be the density of y_t for each t, provide a recursive relationship linking ψ_{t+1} and ψ_t. Your answer should be expressed in terms of these two densities, the density ϕ and the parameters in (2).

2. State a condition under which $\{\psi_t\}$ converges to a unique stationary density ψ^* regardless of y_0.

Solution 7. Possible solutions are as follows:

1. The recursive relationship is

$$\psi_{t+1}(y') = \int \phi\left(\frac{y' - \rho y - b}{c}\right) \frac{1}{c} \psi_t(y)\, \mathrm{d}y$$

   This follows from a change of variable argument given in the text.

2. A sufficient condition is that $|\rho| < 1$.

Question 8. An infinitely lived household has no opportunities to trade and consumes only one good: a kind of fish caught from a local river. The amount of fish caught at time t is denoted by y_t. The process $(y_t)_{t \geq 0}$ is IID with distribution ϕ supported on the integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. An integer number of fish can be stored indefinitely at no cost, although the household can never store more than K units at one time. Let $c_t \in \mathbb{Z}_+$ be current consumption, let $\beta \in (0, 1)$ be a discount factor and let u be a utility function defined on $\mathbb{R}_+$. The household seeks to maximize

$$\mathbb{E} \sum_{t \geq 0} \beta^t u(c_t)$$

Let the number of fish stored at time t be denoted by $s_t \geq 0$. The timing for the problem is that the household makes its time t consumption decision (and consumes) after observing s_t but prior to observing y_{t+1}. The process $(s_t)_{t \geq 0}$ evolves according to

$$s_{t+1} = \min\{s_t - c_t + y_{t+1}, K\}$$

1. Let Σ denote the set of feasible consumption policies for the household. Define this set mathematically. (Consider only stationary Markov policies.)

2. Write down the Bellman equation for this problem in terms of model primitives. Use a summation and ϕ to be explicit about expectations.

Solution 8. Possible solutions are as follows:

1. The set Σ can be defined as

$$\Sigma = \{\sigma \colon \mathbb{K} \to \mathbb{K} \mid 0 \leq \sigma(s) \leq s,\ s \in \mathbb{K}\}$$

   where $\mathbb{K} := \{0, 1, \ldots, K\}$.

2. The Bellman equation is

$$v(s) = \max_{0 \leq c \leq s} \left\{ u(c) + \beta \sum_{y \geq 0} v\left(\min\{s - c + y, K\}\right) \phi(y) \right\}$$

Question 9. Continuing on from question 8,

1. Write down the Bellman operator corresponding to the Bellman equation.
2. Show that the Bellman operator is a contraction of modulus β on $(\mathbb{R}^{\mathbb{K}}, d_\infty)$, where $\mathbb{K} := \{0, 1, \ldots, K\}$.
3. Explain how this fact can be used to calculate an approximately optimal policy.

Solution 9. Possible solutions are as follows:

1. The Bellman operator is

$$Tv(s) = \max_{0 \leq c \leq s} \left\{ u(c) + \beta \sum_{y \geq 0} v\left(\min\{s - c + y, K\}\right) \phi(y) \right\}$$

2. Using the inequality

$$|\sup f - \sup g| \leq \sup |f - g| \tag{3}$$

for bounded real valued function presented in text, we have, for fixed $v, w \in \mathbb{R}^{\mathbb{K}}$ and $s \in \mathbb{K}$,

$$
\begin{aligned}
|Tv(s) - Tw(s)| &\leq \max_{0 \leq c \leq s} \beta \left| \sum_{y \geq 0} \left[v\left(\min\{s - c + y, K\}\right) - w\left(\min\{s - c + y, K\}\right) \right] \phi(y) \right| \\
&\leq \max_{0 \leq c \leq s} \beta \sum_{y \geq 0} \left| v\left(\min\{s - c + y, K\}\right) - w\left(\min\{s - c + y, K\}\right) \right| \phi(y) \\
&\leq \beta \|v - w\|_\infty
\end{aligned}
$$

Here the second inequality is the triangle inequality, while the third is obvious. Taking the supremum over all $s \in \mathbb{K}$ gives

$$\|Tv - Tw\|_\infty \leq \beta \|v - w\|_\infty$$

Since v and w were arbitrary elements of $\mathbb{R}^{\mathbb{K}}$, the proof is now done.

3. By iterating with T from arbitrary $v \in \mathbb{R}^{\mathbb{K}}$ we can compute the value function, at least approximately, and from this approximate value function take the greedy policy. By Bellman's principle of optimality, this policy will be (approximately) optimal.

Question 10. Continuing on from question 8,

1. Show how choosing a σ in Σ also selects a Markov chain for the state process $(s_t)_{t \geq 0}$ associated with this policy. Write down a state space and a stochastic kernel for this Markov policy in terms of model primitives.

2. Explain how the Neumann series lemma can be used to obtain $v_\sigma(s)$, the lifetime value of following policy σ from $t = 0$ when the current state is s. Your solution should involve matrix inversion. Explain why the conditions of the Neumann series lemma are met.

Solution 10. Possible solutions are as follows:

1. Once we fix $\sigma \in \Sigma$, the state process evolves according to the first order Markov process
$$s_{t+1} = \min\{s_t - \sigma(s_t) + y_{t+1}, K\}$$
One way to write the stochastic kernel in terms of model primitives is
$$p_\sigma(s, s') = \sum_{y \geq 0} \mathbb{1}\left\{\min\{s - \sigma(s) + y, K\} = s'\right\} \phi(y)$$

2. As discussed in the lecture slides, the value of following policy σ is given by
$$v_\sigma = \sum_{t \geq 0} \beta^t p_\sigma r_\sigma$$
where $r_\sigma(s) := u(\sigma(s))$ and p_σ is understood as a matrix. By the Neumann series lemma, this infinite sum can also be expressed as
$$v_\sigma = (I - \beta p_\sigma)^{-1} r_\sigma$$
The conditions of that lemma are satisfied because $r(\beta p_\sigma) = \beta r(p_\sigma) = \beta$, since the spectral radius of every stochastic matrix is one.

Question 11. Let $(p_t)_{t \geq 0}$ be the price of an asset. Consider the risk neutral value of a perpetual American call option with strike price K for an infinitely lived investor with discount factor $\beta \in (0, 1)$. (A perpetual American call option gives the right to buy the asset at the agreed strike price in every period and never expires.) At each point in time t, the investor observes the price p_t and either exercises the option or continues to the next period. Exercising the option leads to payoff $p_t - K$. Continuing incurs no cost. In the next period the investor again faces the decision of whether to exercise or continue.

Suppose that $(p_t) \overset{\text{IID}}{\sim} \phi$ where ϕ puts all mass on $\mathbb{R}_+$. The value of the option when $p_t = p$ can be expressed as $v(p)$ where v is a suitable function.

1. Write down the Bellman equation for the value of the option.
2. Write down the corresponding Bellman operator.

Solution 11. Possible solutions are as follows:

1. The Bellman equation is

$$v(p) = \max \left\{ p - K,\ \beta \int v(p')\phi(\mathrm{d}p') \right\}$$

2. The Bellman operator is

$$Tv(p) = \max \left\{ p - K,\ \beta \int v(p')\phi(\mathrm{d}p') \right\}$$

Question 12. Continuing on from question 11, suppose that, in addition to the set of assumptions listed there, the price p_t is bounded above by some constant $M > K$. Let $\mathscr{C}$ be the set of all continous functions on $[0, M]$.

1. Show that the Bellman operator maps $\mathscr{C}$ into itself.
2. Show that the Bellman operator T is a contraction of modulus β on $(\mathscr{C}, d_\infty)$.
3. Does this imply that T has a unique fixed point in $\mathscr{C}$? Why or why not?

Solution 12. Possible solutions are as follows:

1. Fix $v \in \mathscr{C}$. For $0 \leq p \leq M$, we have

$$Tv(p) = \max \left\{ p - K,\ \beta \int v(p')\phi(\mathrm{d}p') \right\}$$

   which is clearly continuous in p.

2. Fix v, w in $\mathscr{C}$ and p in $[0, M]$. From the bound

$$|\alpha \vee x - \alpha \vee y| \leq |x - y| \qquad (\alpha, x, y \in \mathbb{R}) \tag{4}$$

provided in the lectures we have

$$\begin{aligned}
|Tv(p) - Tw(p)| &= \left| \max\left\{ p - K,\ \beta \int v(p')\phi(\mathrm{d}p') \right\} - \max\left\{ p - K,\ \beta \int w(p')\phi(\mathrm{d}p') \right\} \right| \\
&\leq \beta \left| \int [v(p') - w(p')]\phi(p') \right| \\
&\leq \beta \int |v(p') - w(p')| \phi(p') \\
&\leq \beta \|v - w\|_\infty
\end{aligned}$$

Taking the supremum over all $p \in [0, M]$ leads to

$$\|Tv - Tw\|_\infty \leq \beta \|v - w\|_\infty$$

3. Yes, $\mathscr{C}$ is complete under d_∞, since it is a closed subset of the set of bounded functions on $[0, M]$, and the latter is complete. (We are using the fact that continuous functions on compact sets are always bounded.)

Question 13. Continuing on from question 12, under the same set of assumptions, show that v is increasing in p and nonnegative. Provide intuition.

Solution 13. Let $i\mathscr{C}_+$ be the increasing nonnegative functions in $\mathscr{C}$ and pick any $v \in i\mathscr{C}_+$. For $0 \leq p \leq M$, we have

$$Tv(p) = \max\left\{ p - K,\ \beta \int v(p')\phi(\mathrm{d}p') \right\}$$

which is clearly increasing in p. Since v is nonnegative so is $\int v(p')\phi(\mathrm{d}p')$. Therefore $Tv(p)$ is also nonnegative. Hence T is invariant on $i\mathscr{C}_+$. Because weak inequalities are preserved under limits, $i\mathscr{C}_+$ is a closed subset of $\mathscr{C}$, so the fixed point v lies in $i\mathscr{C}_+$. In particular, v is increasing and nonnegative.

Question 14. Continuing on from question 13, under the same set of assumptions,

1. Propose a lower dimensional method for solving for the value of the option, obtained by manipulating the Bellman equation. Your idea should reduce the problem to finding an unknown scalar value.
2. Discuss existence, uniqueness and computation of any endogenous objects in your proposed method.

Solution 14. Possible solutions are as follows:

1. Setting $h := \int v(p')\phi(\mathrm{d}p')$, we can manipulate the Bellman equation to obtain

$$h = \int \max\left\{p' - K,\ \beta h\right\} \phi(\mathrm{d}p')$$

This is an equation in the unknown scalar h that represents continuation value prior to discounting. Once h is found, the value of the option can be obtained at any price p by combining the definition of h with the Bellman equation, yielding

$$v(p) = \max\left\{p - K,\ \beta h\right\} \tag{5}$$

2. Consider the map

$$F(h) = \int \max\left\{p' - K,\ \beta h\right\} \phi(\mathrm{d}p') \qquad (h \geq 0)$$

This map F is a contraction of modulus β on the complete (one dimensional) metric space $\mathbb{R}_+$ paired with Euclidean distance, as follows easily from (4). Hence it has a unique solution that can be computed by iterating with F from any $h \geq 0$.

Question 15. Continuing on from questions 11–14, let $v(p)$ be the value of the option as discussed above, and consider now a shift from the current distribution ϕ for the price process to a new distribution $\hat{\phi}$. Let $\hat{v}(p)$ be the new value of the option.

1. Show that $\hat{v} \geq v$ whenever $\hat{\phi}$ first order stochastically dominates ϕ. Provide intuition.

2. Show that $\hat{v} \geq v$ whenever $\hat{\phi}$ is a mean preserving spread of ϕ. Provide intuition.

Solution 15. Possible solutions are as follows:

1. Let ϕ and $\hat{\phi}$ have the stated properties. The map

$$F(h) = \int \max\left\{p' - K,\ \beta h\right\} \phi(\mathrm{d}p') \qquad (h \geq 0)$$

is globally stable, monotone increasing and shifts up at any given h when ϕ shifts to $\hat{\phi}$. This last statement follows from the fact that the integrand

$$p' \mapsto \max\left\{p' - K,\ \beta h\right\}$$

is increasing in p', along with the definition of first order stochastic dominance.

As a consequence, the fixed point h of F increases when we shift from ϕ to $\hat{\phi}$, and hence so does the value of the option at each price, in light of (5).

The intuition is straightforward: a shift up in the sense of first order stochastic dominance for the asset price distribution increases the likelihood of high prices, which increases the value of a call option.

2. We claim first that $F(h) = \int \max\left\{p' - K,\ \beta h\right\} \phi(\mathrm{d}p')$ increases pointwise with the mean-preserving spread. In other words, for any $h \geq 0$,

$$\int \max\left\{p' - K,\ \beta h\right\} \phi(\mathrm{d}p') \leq \int \max\left\{p' - K,\ \beta h\right\} \hat{\phi}(\mathrm{d}p')$$

By definition, there exists a (p', Z) such that $\mathbb{E}\left[Z \mid p'\right] = 0$, $p' \stackrel{\mathrm{d}}{=} \phi$ and $p' + Z \stackrel{\mathrm{d}}{=} \hat{\phi}$. By this fact and the law of iterated expectations,

$$\begin{aligned}\int \max\left\{p' - K,\ \beta h\right\} \hat{\phi}(\mathrm{d}p') &= \mathbb{E}\ \left[\max\left\{p' + Z - K,\ \beta h\right\}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\max\left\{p' + Z - K,\ \beta h\right\} \,\middle|\, p'\right]\right]\end{aligned}$$

Jensen's inequality combined with the convexity of the max function now produces

$$\int \max \{p' - K, \beta h\} \hat{\phi}(\mathrm{d}p') \geq \mathbb{E} \max \{\mathbb{E} [p' + Z - K \mid p'], \beta h\}$$

Using $\mathbb{E} [p' \mid p'] = p'$ and $\mathbb{E} [Z \mid p'] = 0$ leads to

$$\begin{aligned}\int \max \{p' - K, \beta h\} \hat{\phi}(\mathrm{d}p') &\geq \mathbb{E} \max \{p' - K, \beta h\} \\ &= \int \max \{p' - K, \beta h\} \phi(\mathrm{d}p')\end{aligned}$$

Since h was arbitrary, the function F shifts up pointwise. Since F is monotone increasing and a contraction, the fixed point h of F increases when we shift from ϕ to $\hat{\phi}$, and hence so does the value of the option at each price, in light of (5).

The intuition behind this result is as follows: a mean preserving spread increases the upside risk, while the downside risk is already bounded by the fact that an option is a right to exercise but not an obligation. Hence the value of the option rises.

Question 16. Suppose that time t value v_t of a consumption stream $(c_t)_{t \geq 0}$ is defined recursively by

$$v_t = u(c_t) + \beta \, \mathbb{E}_t v_{t+1}$$

where $\beta \in (0, 1)$ is a discount factor. Assume that $u(c_t) = x_t' U x_t$ for some positive definite $n \times n$ matrix U, where

1. $x_{t+1} = A x_t + C \xi_{t+1}$ in $\mathbb{R}^n$ with x_0 given,
2. $\mathscr{G}_t = \{x_0, \xi_0, \xi_1, \ldots, \xi_t\}$ and
3. $(\xi_t)_{t \geq 1}$ is an $\mathbb{R}^j$-valued zero mean IID sequence with respect to $\mathscr{G}_t$ satisfying $\mathbb{E} [\xi_t \xi_t'] = I$.

Time t conditional expectation is $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \,|\, \mathscr{G}_t]$. Guess that $v_t = v(x_t)$ for some fixed function v. Propose a likely functional form for v and verify your guess. State any conditions you need for existence of a solution. Explain how you would compute any objects for which you do not have analytical expressions. Discuss properties of your solution.

Solution 16. We guess that

$$v(x) = x'Vx + \delta$$

for some positive definite V and scalar δ. Substituting $v_t = x_t'Vx_t + \delta$ and $u(c_t) = x_t'Ux_t$ into $v_t = u(c_t) + \beta\mathbb{E}_t v_{t+1}$ gives

$$\begin{aligned} x_t'Vx_t + \delta &= x_t'Ux_t + \beta\mathbb{E}_t[x_{t+1}'Vx_{t+1} + \delta] \\ &= x_t'Ux_t + \beta x_t'A'VAx_t + \beta\,\mathrm{trace}(C'VC) + \beta\delta \end{aligned}$$

Thus, we seek a pair $V \in \mathcal{M}(n \times n)$, $\delta \in \mathbb{R}$ such that

$$x'Vx + \delta = x'Ux + \beta x'A'VAx + \beta\,\mathrm{trace}(C'VC) + \beta\delta$$

for any $x \in \mathbb{R}^n$. If V^* solves

$$V = U + \beta A'VA \tag{6}$$

and

$$\delta^* := \frac{\beta}{1-\beta}\,\mathrm{trace}(C'V^*C)$$

then V^*, δ^* solves the above equation for any x, as can be shown with a small amount of algebra. The only issue is whether there exists a $V \in \mathcal{M}(n \times n)$ that solves (6). If we write this equality as the discrete Lyapunov equation

$$V = \Lambda'V\Lambda + U$$

where $\Lambda := \sqrt{\beta}A$, then V^* is the fixed point of operator ℓ defined by

$$\ell V = \Lambda'V\Lambda + U$$

We know that ℓ is globally stable when $r(\Lambda) < 1$, or equivalently, that $r(A) \leq 1/\sqrt{\beta}$. Under this condition, there is a unique solution to (6), and we can compute by iterating with ℓ from any initial $V_0 \in \mathcal{M}(n \times n)$.

The solution V^* is positive semidefinite, as can be established by noting that the set of positive semidefinite matrices is closed in the set of all matrices under the usual norm topology, and, in addition, ℓ is invariant on the set of positive semidefinite matrices.