Economic Dynamics: Theory and Computation

Second Edition

Additional Exercises

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(Solutions available on request)

PRELIMINARIES

The questions below use the material in the textbook plus the following two additional results.

Theorem 1 (Neumann Series Lemma)**.** *Let A be $n \times n$, let b be $n \times 1$ and let I be the $n \times n$ identity matrix. If $r(A) < 1$, then $I - A$ is nonsingular and $x = Ax + b$ has the unique solution*

$$x^* = (I - A)^{-1} b = \sum_{k \geq 0} A^k b.$$

In the theorem above, $r(A)$ is the spectral radius of A, defined by

$$r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} \tag{1}$$

Here $|\lambda|$ indicates the modulus of the complex number λ.

In the next result, we take S and T to be self-maps on $M \subset \mathbb{R}^n$ and let $\leq$ be the pointwise partial order on $\mathbb{R}^n$. We say that T dominates S on M if $Sx \leq Tx$ for all $x \in M$.

Theorem 2. *If T dominates S on M and, in addition, T is monotone increasing and globally stable on M, then its unique fixed point dominates any fixed point of S.*

Also, for a given pair of distributions ϕ, ψ on $\mathbb{R}$, we say that ψ *first order stochastically dominates* ϕ if

$$\int u(x)\phi(\mathrm{d}x) \leq \int u(x)\psi(\mathrm{d}x) \;\text{ for every increasing bounded function } u \colon \mathbb{R} \to \mathbb{R}$$

Finally, we say that ψ is a *mean-preserving spread* of ϕ if there exists a pair of random variables (Y, Z) such that

$$\mathbb{E}\,[Z \,|\, Y] = 0, \quad Y \stackrel{\mathrm{d}}{=} \phi \quad \text{and} \;\; Y + Z \stackrel{\mathrm{d}}{=} \psi$$

In other words ψ is a mean-preserving spread of ϕ if it adds noise without changing the mean.

Questions

Question 1. Let $D = [0, 1]$ and let $f \colon D \to \mathbb{R}$ be defined by $f(x) = 1 - |x - 0.5|$. List the maximizers and minimizers of f on D, if any. Explain your reasoning.

Question 2. Give an example of a function f from $\mathbb{R}$ to $\mathbb{R}$ that has neither a maximizer nor a minimizer. Explain why the function has this property.

Question 3. Let $f \colon D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$. Suppose that a and b are both maximizers of f on D. Is it necessarily true that $f(a) = f(b)$? Why or why not?

Question 4. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|$. Is f a bijection? Why or why not?

Question 5. An infinitely lived worker can either be unemployed (state 0) or employed (state 1). If the worker is currently unemployed, then she remains unemployed with probability α and becomes employed with probability $1 - \alpha$. If the worker is currently employed, then she remains employed with probability β and becomes unemployed with probability $1 - \beta$. Let $X_t \in \{0, 1\}$ denote the current state of the worker.

1. Write down the stochastic kernel p for the Markov chain $(X_t)_{t \geq 0}$ as a matrix.
2. Does p have a stationary distribution? Why or why not?
3. Give a set of conditions on α, β such that p has exactly one stationary distribution. Connect your condition to the discussion of Markov chains in the lecture slides or course notes.

Question 6. Consider a worker whose log income obeys

$$y_{t+1} = \rho y_t + b + c\xi_{t+1} \tag{2}$$

where y_0 is a constant and $(\xi_t)_{t \geq 1}$ is an IID stochastic process with $\mathbb{E}\,\xi_t = 0$ and $\mathbb{E}\,\xi_t^2 = 1$. Let $\mu_t := \mathbb{E}\,y_t$ and let σ_t^2 be the variance of y_t.

1. Provide a recursive relationship linking μ_{t+1} and μ_t.
2. State a condition under which $\{\mu_t\}$ converges and provide an expression for the limit.
3. Provide a recursive relationship linking σ_{t+1}^2 and σ_t^2.
4. State a condition under which $\{\sigma_t^2\}$ converges and provide an expression for the limit.

Question 7. Continuing on from question 6, suppose in addition that $c > 0$ and each ξ_t has density ϕ on $\mathbb{R}$.

1. Letting ψ_t be the density of y_t for each t, provide a recursive relationship linking ψ_{t+1} and ψ_t. Your answer should be expressed in terms of these two densities, the density ϕ and the parameters in (2).
2. State a condition under which $\{\psi_t\}$ converges to a unique stationary density ψ^* regardless of y_0.

Question 8. An infinitely lived household has no opportunities to trade and consumes only one good: a kind of fish caught from a local river. The amount of fish caught at time t is denoted by y_t. The process $(y_t)_{t \geq 0}$ is IID with distribution ϕ supported on the integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. An integer number of fish can be stored indefinitely at no cost, although the household can never store more than K units at one time. Let $c_t \in \mathbb{Z}_+$ be current consumption, let $\beta \in (0,1)$ be a discount factor and let u be a utility function defined on $\mathbb{R}_+$. The household seeks to maximize

$$\mathbb{E} \sum_{t \geq 0} \beta^t u(c_t)$$

Let the number of fish stored at time t be denoted by $s_t \geq 0$. The timing for the problem is that the household makes its time t consumption decision (and consumes) after observing s_t but prior to observing y_{t+1}. The process $(s_t)_{t \geq 0}$ evolves according to

$$s_{t+1} = \min\{s_t - c_t + y_{t+1}, K\}$$

1. Let Σ denote the set of feasible consumption policies for the household. Define this set mathematically. (Consider only stationary Markov policies.)
2. Write down the Bellman equation for this problem in terms of model primitives. Use a summation and ϕ to be explicit about expectations.

Question 9. Continuing on from question 8,

1. Write down the Bellman operator corresponding to the Bellman equation.
2. Show that the Bellman operator is a contraction of modulus β on $(\mathbb{R}^{\mathbb{K}}, d_\infty)$, where $\mathbb{K} := \{0, 1, \ldots, K\}$.
3. Explain how this fact can be used to calculate an approximately optimal policy.

Question 10. Continuing on from question 8,

1. Show how choosing a σ in Σ also selects a Markov chain for the state process $(s_t)_{t \geq 0}$ associated with this policy. Write down a state space and a stochastic kernel for this Markov policy in terms of model primitives.

2. Explain how the Neumann series lemma can be used to obtain $v_\sigma(s)$, the lifetime value of following policy σ from $t = 0$ when the current state is s. Your solution should involve matrix inversion. Explain why the conditions of the Neumann series lemma are met.

Question 11. Let $(p_t)_{t \geq 0}$ be the price of an asset. Consider the risk neutral value of a perpetual American call option with strike price K for an infinitely lived investor with discount factor $\beta \in (0, 1)$. (A perpetual American call option gives the right to buy the asset at the agreed strike price in every period and never expires.) At each point in time t, the investor observes the price p_t and either exercises the option or continues to the next period. Exercising the option leads to payoff $p_t - K$. Continuing incurs no cost. In the next period the investor again faces the decision of whether to exercise or continue.

Suppose that $(p_t) \overset{\text{IID}}{\sim} \phi$ where ϕ puts all mass on $\mathbb{R}_+$. The value of the option when $p_t = p$ can be expressed as $v(p)$ where v is a suitable function.

1. Write down the Bellman equation for the value of the option.
2. Write down the corresponding Bellman operator.

Question 12. Continuing on from question 11, suppose that, in addition to the set of assumptions listed there, the price p_t is bounded above by some constant $M > K$. Let $\mathscr{C}$ be the set of all continous functions on $[0, M]$.

1. Show that the Bellman operator maps $\mathscr{C}$ into itself.
2. Show that the Bellman operator T is a contraction of modulus β on $(\mathscr{C}, d_\infty)$.
3. Does this imply that T has a unique fixed point in $\mathscr{C}$? Why or why not?

Question 13. Continuing on from question 12, under the same set of assumptions, show that v is increasing in p and nonnegative. Provide intuition.

Question 14. Continuing on from question 13, under the same set of assumptions,

1. Propose a lower dimensional method for solving for the value of the option, obtained by manipulating the Bellman equation. Your idea should reduce the problem to finding an unknown scalar value.
2. Discuss existence, uniqueness and computation of any endogenous objects in your proposed method.

Question 15. Continuing on from questions 11–14, let $v(p)$ be the value of the option as discussed above, and consider now a shift from the current distribution ϕ for the price process to a new distribution $\hat{\phi}$. Let $\hat{v}(p)$ be the new value of the option.

1. Show that $\hat{v} \geq v$ whenever $\hat{\phi}$ first order stochastically dominates ϕ. Provide intuition.
2. Show that $\hat{v} \geq v$ whenever $\hat{\phi}$ is a mean preserving spread of ϕ. Provide intuition.

Question 16. Suppose that time t value v_t of a consumption stream $(c_t)_{t \geq 0}$ is defined recursively by

$$v_t = u(c_t) + \beta \, \mathbb{E}_t v_{t+1}$$

where $\beta \in (0, 1)$ is a discount factor. Assume that $u(c_t) = x_t' U x_t$ for some positive definite $n \times n$ matrix U, where

1. $x_{t+1} = A x_t + C \xi_{t+1}$ in $\mathbb{R}^n$ with x_0 given,

2. $\mathscr{G}_t = \{x_0, \xi_0, \xi_1, \ldots, \xi_t\}$ and

3. $(\xi_t)_{t \geq 1}$ is an $\mathbb{R}^j$-valued zero mean IID sequence with respect to $\mathscr{G}_t$ satisfying $\mathbb{E}[\xi_t \xi_t'] = I$.

Time t conditional expectation is $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathscr{G}_t]$. Guess that $v_t = v(x_t)$ for some fixed function v. Propose a likely functional form for v and verify your guess. State any conditions you need for existence of a solution. Explain how you would compute any objects for which you do not have analytical expressions. Discuss properties of your solution.
