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Abstract For Markovian economic models, long-run equilibria are typically identified with the stationary (invariant) distributions generated by the model. In this paper we provide new sufficient conditions for continuity in the map from parameters to these equilibria. Several existing results are shown to be special cases of our theorem.
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Parametric Continuity of Stationary Distributions

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1 Introduction

Let X_t be a vector of endogenous and exogenous variables, jointly following a Markov process generated by some underlying model. In economic dynamics, one frequently considers situations where the sequence $(X_t)_{t=0}^{\infty}$ is stationary. For example, Brock and Mirman (1972) famously proved that the stochastic optimal growth model admits a stationary process, and that every process is in fact asymptotically stationary.

In the Markov case, stationarity reduces to the existence of a "stationary distribution" μ, such that if X_t has law μ, then so does X_{t+j} for all $j \in \mathbb{N}$. If such a μ exists then it naturally becomes a focus of equilibrium analysis. For example, if μ is also unique and has certain stability properties, then a law of large numbers (ergodicity) result also holds, in which case sample moments from the series $(X_t)_{t=0}^{\infty}$ can be identified with integrals of the relevant functions with respect to the stationary distribution μ.

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Typically, the underlying laws which drive the process $(X_t)_{t=0}^{\infty}$ depend on a vector of parameters, which may for example be policy instruments, or regression coefficients to be estimated from the data. In this case the parameters themselves determine the stationary distribution. Our paper investigates conditions under which the functional relationship between parameters and stationary distributions is continuous.[1]

The study of how stationary distributions vary with the parameters is a stochastic analogue of standard comparative dynamics and has many applications. A typical example is found in the Simulated Moments Estimator of Duffie and Singleton (1993), consistency of which requires continuity of stationary distributions in the unknown parameters.[2] Another example is in computational economics, where perturbed models may be easier to solve numerically than the original, but solutions of the perturbed system must be close to those of the original model as the level of perturbation becomes small. An application of this type is discussed in Section 5.

Within economics, perhaps the best known result in this area (parametric continuity for stochastic equilibria) is a theorem which appears in Stokey, Lucas and Prescott (1989, Theorem 12.13) and is apparently due to R.E. Manuelli. The result pertains to Markov models on a compact state space. Another result in the literature is that of Stenflo (2001), who proves parametric continuity for noncompact state spaces when the transition rule is contracting on average.

In this paper, we use Berge's Theorem of the Maximum to provide a new parametric continuity result. The basic idea is as follows: Stationary distributions can be identified as the fixed points of a certain operator P_θ mapping distributions into distributions, where $\theta \in \Theta$ is a parameter. If we can furnish a metric ϱ on the space of distributions, then the function $F(\theta, \mu) := -\varrho(\mu, P_\theta(\mu))$ is zero if and only if μ is stationary given θ. In fact, providing that at least one stationary distribution exists for each θ, it is clear that the set of stationary distributions and the set of maximizers of $\mu \mapsto F(\theta, \mu)$ coincide. When Berge's conditions are satisfied, his Theorem of the Maximum tells us precisely when the dependence of these maximizers on the parameters will be continuous.

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[1] A related question is parametric *monotonicity* of stationary distributions. See, for example, Huggett (2003), or Mirman, Morand and Reffett (2005).

[2] See in particular, Duffie and Singleton (1993, Section 4.3, Assumption 1).

We then show that both the result of Manuelli and the parametric continuity result of Stenflo are in fact special cases of our theorem.[3] We also provide a new result (Proposition 2) which is another special case of the main theorem, and should prove useful in applications. This claim is illustrated using two examples. The first is a rational expectations pricing problem, and the second is a simple growth model.

Readers interested in applying the techniques in this paper rather than studying the theory should go directly to Section 5, and consult Proposition 2. The two examples are intended to provide guidance on how to verify the assumptions of the proposition.

2 Set Up

Let $\mathscr{P}(S)$ be the collection of probabilities on $(S, \mathscr{B}(S))$, where S is any separable, completely metrizable topological space, and $\mathscr{B}(S)$ is its Borel sets. Let $\mathscr{M}(S)$ be the linear space of finite signed measures on $(S, \mathscr{B}(S))$, and let $bC(S)$ be the bounded continuous real valued functions on S. For $\mu \in \mathscr{M}(S)$ and $h \in bC(S)$ we use the symmetric notation $\langle \mu, h\rangle = \langle h, \mu\rangle$ to denote $\int_S h d\mu$. Let $w(\mathscr{M}(S), bC(S))$ be the weak topology on $\mathscr{M}(S)$ generated by the set of linear functionals $\mu \mapsto \langle \mu, h\rangle$, $h \in bC(S)$, in the usual way (see, e.g., Stokey, Lucas and Prescott, Chapter 12), and let $w(\mathscr{P}(S), bC(S))$ be the relative topology on $\mathscr{P}(S)$.

Below we make use of the following well-known Skorohod–Dudley representation theorem: If $(\mu_n)_{n=1}^{\infty}$ is a sequence in $\mathscr{P}(S)$ and $\mu_n \to \mu \in \mathscr{P}(S)$ in the topology $w(\mathscr{P}(S), bC(S))$, then there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ supporting S-valued random variables $(X_n)_{n=1}^{\infty}$ and X with X_n (resp., X) having distribution μ_n (resp., μ) and $X_n \to X$ $\mathbb{P}$-almost surely.[4]

We also make use of the fact that $w(\mathscr{P}(S), bC(S))$ is metrizable. In particular, the Fortet-Mourier metrization of $w(\mathscr{P}(S), bC(S))$ is defined as follows: Let d be any distance function which metrizes the topology on S. Let

[3] A caveat is that for the problem Stenflo considers we require that the closed and bounded subsets of the state space are precisely the compact sets—as is the case, for example, with finite-dimensional Euclidean vector space. By contrast, Stenflo's results hold in any completely metrizable topological space.

[4] See, for example, Dudley (2002, Theorem 11.7.2).

$bL(S,d)$ be the collection of bounded Lipschitz functions on (S,d). This space is given the norm

$$\|h\|_{bL} := \sup_{x \in S} |h(x)| + \sup_{x \neq y} \frac{|h(x) - h(y)|}{d(x,y)}. \tag{1}$$

Now set $\varrho_{FM}(\mu, \nu) := \sup |\langle \mu, h \rangle - \langle \nu, h \rangle|$, where the supremum is over all $h \in bL(S,d)$ with $\|h\|_{bL} \leq 1$. The function ϱ_{FM} so defined is known to metrize $w(\mathscr{P}(S), bC(S))$.[5]

A stochastic kernel (or transition probability function) P on S is a family of probability measures

$$P(x, dy) \in \mathscr{P}(S), \quad x \in S$$

such that $x \mapsto P(x, B)$ is Borel measurable for each $B \in \mathscr{B}(S)$.

We set $Ph(x) := \int_S h(y) P(x, dy)$ for real valued h on S where this integral is defined. In addition, for $\mu \in \mathscr{M}(S)$, we write μP for the element of $\mathscr{M}(S)$ defined by $(\mu P)(B) := \int P(x, B) \mu(dx)$. Thus, P can be regarded as an operator which acts on functions to the right and measures to the left.[6] It can easily be shown that $h \mapsto Ph$ is a positive (i.e., increasing) linear operator on $bC(S)$, as is $\mu \mapsto \mu P$ on $\mathscr{M}(S)$. Clearly $P\mathbb{1}_S = \mathbb{1}_S$. Also, $\langle \mu P, h \rangle = \langle Ph, \mu \rangle$ for all $h \in bC(S)$ and all $\mu \in \mathscr{P}(S)$.[7] Let P^t denote t compositions of P with itself.

For $x \in S$ we use δ_x to denote the probability with unit mass on x. It is well-known that $\delta_x P^t$ is the marginal distribution of X_t given that $X_0 \equiv x \in S$, and $(X_t)_{t=0}^\infty$ follows the Markov process defined by P; while $P^t h(x)$ is the expectation of $h(X_t)$ conditional on $X_0 \equiv x$. The reader is referred to Stokey, Lucas and Prescott (1989, p. 213) for further discussion.

Given P, a stationary or invariant distribution is a $\mu \in \mathscr{P}(S)$ such that $\mu P = \mu$. A function $V \colon S \to [0, \infty)$ is called a Lyapunov function (or simply Lyapunov) if it is continuous and all sublevel sets

$$C_{V,a} := \{x \in S : V(x) \leq a\}, \quad a \in \mathbb{R}$$

[5] For a proof see, for example, Dudley (2002, Theorem 11.3.3). In fact it is not difficult to prove the same using the Skorohod–Dudley representation theorem.

[6] Our notation is quite standard. See, for example, the classic monograph of Meyn and Tweedie (1993).

[7] In other words, the two operators are adjoint. See Stokey, Lucas and Prescott (1989, Theorem 8.3).

are compact.[8] Let $\mathscr{L}(S)$ be the set of Lyapunov functions on S. Finally, a subset Q of $\mathscr{P}(S)$ is called tight if, for all $\varepsilon > 0$, there is a compact $K \subset S$ such that $\sup_{\mu \in Q} \mu(S \setminus K) \leq \varepsilon$.

3 Results

Our starting point is a parameter space Θ and a family of stochastic kernels $\{P_\theta : \theta \in \Theta\}$. Here Θ is a metrizable topological space. Let N denote any subset of Θ. Define $\Lambda(\theta) := \{\mu \in \mathscr{P}(S) : \mu = \mu P_\theta\}$.

Assumption 1 $N \times \mathscr{P}(S) \ni (\theta, \mu) \mapsto \mu P_\theta \in \mathscr{P}(S)$ is continuous.[9]

Assumption 1 is a continuity assumption on the primitives, without which there can be little hope of general continuity results for solutions. Below we develop versions of this condition which are easier to verify in applications. It should also be noted that the continuity in Assumption 1 helps to ensure existence of stationary distributions via Brouwer–Schauder type fixed point arguments.

Assumption 2 For each $\theta \in N$, there is a $V \in \mathscr{L}(S)$ and $x \in S$ such that $\liminf_{t \to \infty} P_\theta^t V(x) < \infty$.

As discussed above, the term $P_\theta^t V(x)$ can be interpreted as the expectation of $V(X_t)$, where $(X_t)_{t=0}^\infty$ is the Markov process starting at initial condition x and driven by stochastic kernel P_θ. Boundedness of this expectation in the limit helps to contain probability mass to regions where V is relatively small. In turn, this bounding of probability mass is closely connected to stability. For example, the following existence result is immediate from Meyn and Tweedie (1993, Proposition 12.1.3).

[8] Loosely speaking, when S is not itself compact, V must get large towards the "edges" of S. A classic example of a Lyapunov function is the Euclidean (in fact any) norm in $\mathbb{R}^n$. For our purposes it is useful to note that if S is compact then every continuous nonnegative function on S is Lyapunov. Alternatively, if d metrizes the topology on S and the closed bounded subsets of (S, d) are compact, then $V(x) = d(x, x_0)$ is Lyapunov for each $x_0 \in S$.

[9] Unless otherwise stated, all topological notions concerning $\mathscr{P}(S)$ refer to $w(\mathscr{P}(S), bC(S))$. Also, product spaces are given the product topology.

Lemma 1 *If Assumptions 1 and 2 hold, then the set of stationary distributions $\Lambda(\theta)$ is nonempty for all $\theta \in N$.*

Parametric continuity is a classic problem of interchanging orders of limits, for which a degree of uniformity is usually required. The next assumption implies a kind of uniform compactness. In stating it we adopt the following notation: For $W \in \mathscr{L}(S)$ and $M \in \mathbb{N}$, let $\Gamma(W, M)$ be the set of all $\mu \in \mathscr{P}(S)$ satisfying $\int W d\mu \leq M$.

Assumption 3 There exists a $W \in \mathscr{L}(S)$ and an $M \in \mathbb{N}$ such that $\Lambda(\theta) \subset \Gamma(W, M)$ for all $\theta \in N$.[10]

We can now present our main result:

Theorem 1 *If Assumptions 1–3 hold for some $N \subset \Theta$, then the correspondence $\theta \mapsto \Lambda(\theta)$ is nonempty, compact valued, and upper hemicontinuous on N.*

Proof Define $F(\theta, \mu) := -\varrho_{FM}(\mu, \mu P_\theta)$.[11] Taking W and M as given in Assumption 3, set $H(\theta) := \operatorname{argmax}_{\mu \in \Gamma(W,M)} F(\theta, \mu)$. The theorem will be verified if we can show that (i) $\theta \mapsto H(\theta)$ is nonempty, compact valued and upper hemicontinuous on N, and (ii) that $H(\theta) = \Lambda(\theta)$ on N.

Claim (i) is immediate from Berge's Theorem of the Maximum (Aliprantis and Border, 1999, p. 539), as the function F is continuous on $N \times \mathscr{P}(S)$ by Assumption 1; and $\Gamma(W, M)$ is compact (see the comments in footnote 10) and nonempty (by Lemma 1 and Assumption 3). Regarding claim (ii), fix $\theta \in N$. If $\mu \in \Lambda(\theta)$, then $\mu \in \Gamma(W, M)$ by Assumption 3, and $F(\theta, \mu) = 0$. Hence μ is a maximizer, and by definition an element of $H(\theta)$. Conversely, if $\mu \in H(\theta)$, then $F(\theta, \mu) \geq 0$ and hence $\mu \in \Lambda(\theta)$, as $\Lambda(\theta) \subset \Gamma(W, M)$ by assumption, and on $\Lambda(\theta)$ the function F attains zero.

Remark 1 In particular, if there is a unique fixed point μ_θ for each $\theta \in N$, then $\theta \mapsto \mu_\theta$ is continuous on N.

[10] In applying Assumptions 2 and 3 we make use of the following result: If $V \in \mathscr{L}(S)$, $M \in \mathbb{N}$, and $Q \subset \mathscr{P}(S)$ with $\sup_{\mu \in Q} \int V d\mu \leq M$ then Q is tight. (The proof is not difficult. See Meyn and Tweedie, 1993, Lemma D.5.3.) The closure of Q is then $w(\mathscr{P}(S), bC(S))$-compact by Prohorov's theorem.

[11] The metric ϱ_{FM} was defined above. In fact any distance function which metrizes our weak topology on $\mathscr{P}(S)$ will do.

4 Existing Applications

In this section we show how some seemingly unrelated existing results can be derived from Theorem 1.

4.1 Compact State

First, consider the compact state space result of Stokey, Lucas and Prescott (1989, Theorem 12.13), which is apparently due to R.E. Manuelli:

Theorem 2 *Let S be compact. If Assumption 1 holds for some $N \subset \Theta$ and $\Lambda(\theta)$ is single valued, then $\theta \mapsto \Lambda(\theta)$ is continuous on N.*

This result is immediate from Theorem 1: Set $V = W = 0$ everywhere on S and let $M = 0$ in Assumptions 2 and 3.

Even though this theorem is quite straightforward, it is not always easy to check Assumption 1 in applications. For example, the joint continuity of $(\theta, \mu) \mapsto \mu P_\theta$ is more difficult to check that the requirement that $\mu \mapsto \mu P_\theta$ and $\theta \mapsto \mu P_\theta$ are continuous for each θ and μ respectively. Moreover, the immediate object of interest in economic studies is usually a stochastic difference equation, rather than a stochastic kernel. Finally, in much of applied macroeconomics the state space is not compact. Below we discuss results which address some of these concerns.

4.2 Average Contractions

In this section we review the results of Stenflo (2001, Theorem 2). Suppose that $S = (S, d)$ is boundedly compact.[12] In this case it turns out that his parametric continuity theorem is also a special case of Theorem 1.[13] To state

[12] A metric space is called boundedly compact if all the closed balls are compact. The finite dimensional vector spaces are typical examples. We need bounded compactness of S to ensure that $x \mapsto d(x, x_0)$ is Lyapunov on S for all $x_0 \in S$.

[13] It should be noted, however, that Stenflo obtains rates of convergence. Rates are useful for deriving error bounds in computational problems. In contrast, Theorem 1 cannot be used to derive rates.

his theorem, let $(Z, \mathscr{Z})$ be an arbitrary measurable space, and let $\mathscr{P}(Z)$ be the probabilities on $(Z, \mathscr{Z})$. Stenflo considers the stochastic recursive model

$$X_{t+1} = T_\theta(X_t, \xi_{t+1}), \text{ where } \xi_t \sim \psi_\theta \in \mathscr{P}(Z), \ \forall t \in \mathbb{N}. \tag{2}$$

Here T_θ is a measurable function sending $S \times Z \to S$ for each $\theta \in \Theta$, and $(\xi_t)_{t=1}^\infty$ is an independent sequence, all with distribution ψ_θ. For $x \in S$ and $B \in \mathscr{B}(S)$ we set $P_\theta(x, B) := \psi_\theta\{z \in Z : T_\theta(x, z) \in B\}$. Finally, let e metrize the topology on Θ.

Stenflo makes the following assumptions. (As before, N is an arbitrary subset of Θ.)

Assumption 4 There exists a $\lambda \in (0, 1)$ such that, $\forall\, \theta \in N$,

$$\int d(T_\theta(x, z), T_\theta(x', z))\psi_\theta(dz) \leq \lambda d(x, x'), \quad \forall\, x, x' \in S.$$

Assumption 5 There exists an $x_0 \in S$ such that

$$L := \sup_{\theta \in N} \int d(T_\theta(x_0, z), x_0)\psi_\theta(dz) < \infty.$$

It is known (see, e.g., Stenflo, 2001, Theorem 1) that

Lemma 2 *If Assumptions 4 and 5 hold, then P_θ has a unique stationary distribution $\mu_\theta \in \mathscr{P}(S)$ for each $\theta \in N$. Moreover, for each $x \in S$ and $\theta \in N$ we have $\delta_x P_\theta^t \to \mu_\theta$ as $t \to \infty$.*

To derive parametric continuity he requires in addition:

Assumption 6 There exists a function δ mapping $[0, \infty)$ to itself such that $\delta(x) \to 0$ when $x \to 0$, and

$$\sup_{z \in Z} \sup_{x \in S} d(T_\theta(x, z), T_{\theta'}(x, z)) \leq \delta(e(\theta, \theta')), \quad \forall\, \theta, \theta' \in N.$$

Assumption 7 The map $N \ni \theta \mapsto \psi_\theta \in \mathscr{P}(Z)$ is continuous with respect to the total variation norm topology on $\mathscr{P}(Z)$.

Theorem 3 (Stenflo) *Let μ_θ be as in Lemma 2. If Assumptions 4–7 all hold, then $\theta \to \mu_\theta$ is continuous on N.*

When S is boundedly compact this turns out to be a special case of Theorem 1:

Proposition 1 *If S is boundedly compact, then Assumptions 4—7 imply Assumptions 1–3, with $V(x) = W(x) = d(x, x_0)$ and $M = L/(1-\lambda)$.*

The proof of the proposition is given in the appendix.

5 A Further Application

Next we develop a new application of Theorem 1, which extends Stenflo's results in Section 4.2 and is intended to be useful in economic applications. Let S be a separable and completely metrizable topological space (unlike the previous section, S need not be boundedly compact), and let $(Z, \mathscr{Z})$ again be a measure space. Consider once more the model

$$X_{t+1} = T_\theta(X_t, \xi_{t+1}), \quad \text{where} \quad \xi_t \sim \psi_\theta \in \mathscr{P}(Z), \quad \forall t \in \mathbb{N}.$$

Here $T_\theta \colon S \times Z \to S$ is measurable, $(\xi_t)_{t=1}^\infty$ are independent and identically distributed, N is an arbitrary subset of the parameter space (Θ, e) where e is a metric on Θ, and $P_\theta(x, B) := \psi_\theta\{z \in Z : T_\theta(x, z) \in B\}$ is the stochastic kernel corresponding to this model.

First, we wish to weaken Assumption 6, which is too restrictive in some applications (see below). The following condition is clearly weaker:

Assumption 8 The map $N \ni \theta \mapsto T_\theta(x, z) \in S$ is continuous for each pair $(x, z) \in S \times Z$.

We wish also to relax Assumption 4, which requires that the law of motion is contracting on average. For example, if we take $S = Z = \mathbb{R}$, $d(x, y) = |x - y|$, and law of motion $X_{t+1} = g_\theta(X_t) + \xi_{t+1}$, then Assumption 4 requires that g_θ has slope with absolute value strictly less than one everywhere on $\mathbb{R}$, uniformly over all $\theta \in N$. Such a requirement is rather strict. Instead consider

Assumption 9 For each compact $C \subset S$, there is a $K < \infty$ with

$$\int d(T_\theta(x, z), T_\theta(x', z))\psi_\theta(dz) \le K d(x, x'), \quad \forall x, x' \in C, \ \forall \theta \in N.$$

In essence, this is a local Lipschitz assumption.

Next we add drift with respect to a Lyapunov function, which has the effect of shifting probability mass towards areas of the state space where the Lyapunov function is small:

Assumption 10 There exists a $V \in \mathscr{L}(S)$, $\lambda \in (0,1)$ and $L \in [0,\infty)$ such that, $\forall \theta \in N$,

$$P_\theta V(x) := \int V(T_\theta(x,z))\psi_\theta(dz) \leq \lambda V(x) + L, \quad \forall x \in S.$$

Finally, an assumption is necessary on the continuity of $\theta \mapsto \psi_\theta$:

Assumption 11 Either $\theta \mapsto \psi_\theta$ is continuous in total variation norm, or (i) $\theta \mapsto \psi_\theta$ is continuous in $w(\mathscr{P}(S), bC(S))$, (ii) Z is a separable and completely metrizable topological space, and (iii) for each $x \in S$ and compact $C \subset Z$, there is a $J < \infty$ s.t.

$$d(T_\theta(x,z), T_\theta(x,z')) \leq J d(z,z'), \quad \forall z, z' \in C, \ \forall \theta \in N.$$

Regarding Assumption 11, note that for continuity in total variation norm, Z need not be a topological space. Continuity in total variation subsumes the important special case that the distribution ψ_θ does not in fact depend on θ. The second alternative requires a local Lipschitz property on $z \mapsto T_\theta(x,z)$.[14]

Under these assumptions we have the following result, the proof of which is given in the appendix:

Proposition 2 *If Assumptions 8–11 hold, then $\Lambda(\theta)$ is nonempty for each $\theta \in N$. If $\Lambda(\theta) = \{\mu_\theta\}$, then $\theta \mapsto \mu_\theta$ is continuous on N.*

We give two applications of Proposition 2.

Example 1. Consider the following (slightly simplified) version of the speculative storage model treated in Bobenrieth, Bobenrieth and Wright (2006); hereafter BBW. Production of a given commodity is identified with an IID sequence of "harvests" $(\xi_t)_{t=1}^\infty$. Agents consist of a continuum of identical consumers and speculators (storers), each of measure one. Consumers have

[14] For simplicity we are using the same notation d for the metric on S and the metric on Z, although they may be different.

inverse demand curve f, the properties of which are given below. Demand for the commodity in each period is the sum of demand by consumers and demand by speculators, and supply is the sum of the harvest and λq, where $\lambda \in (0,1)$ parameterizes depreciation from period to period, and q is the quantity stored by speculators in the previous period.

It is well-known (interested readers should consult BBW or Samuelson, 1971) that an equilibrium system for prices $(p_t)_{t=0}^{\infty}$, consumption $(c_t)_{t=0}^{\infty}$ and storage $(q_t)_{t=0}^{\infty}$ can be found by solving the problem

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \qquad \text{s.t. } X_{t+1} = \lambda q_t + \xi_{t+1}, \quad q_t + c_t = X_t,$$

of a representative agent, where X_t is supply at time t, the utility function u is defined by $\int_0^x f(z)dz = u(x)$ and β is the inverse of the gross interest rate.

Following BBW, we assume that u is twice differentiable, strictly increasing, strictly concave, bounded, and satisfies the interiority condition $\lim_{c \downarrow 0} u'(c) = \infty$; and that the distribution ψ of ξ_t has finite first moment and consists of both singular and absolutely continuous components (with respect to Lebesgue measure). That is, $\psi = \alpha\psi_s + (1-\alpha)\psi_c$, $0 < \alpha < 1$. BBW assume that the singular component ψ_s has an atom at zero. To ease the exposition we assume here further than $\psi_s = \delta_0$.

Using standard arguments, BBW show that the representative agent's optimization problem is solved by a unique storage (carryover) function q, which maps current supply X_t to storage quantity $q(X_t)$, and induces law of motion

$$X_{t+1} = \lambda q(X_t) + \xi_{t+1}, \quad \xi_t \sim \psi. \tag{3}$$

The map q is continuous and increasing on $[0,\infty)$. The corresponding consumption function $c(x) := x - q(x)$ is also increasing. Equilibrium prices are given by

$$p_t = p(X_t), \quad p := u' \circ c = f \circ c.$$

A unique stationary distribution μ exists for $(X_t)_{t=0}^{\infty}$ given by (3).[15] The stationary distribution for prices is $\mu \circ c^{-1} \circ f^{-1}$.

In their paper, BBW discuss computation of the stationary distribution for prices. This exercise is complicated by the atom at zero in the shock distribution, which in turn causes unboundedness of the equilibrium pricing

[15] A simple proof of existence and uniqueness can be obtained from Lemma 2.

functional (see BBW for further details). To avoid this difficulty, BBW consider a perturbed version of the same model, where the shock has no atom at zero. In what follows the perturbation is implemented by setting $\psi_\theta := \alpha\delta_\theta + (1-\alpha)\psi_c$, where $\theta \geq 0$.

For this perturbation the corresponding representative agent's optimization problem is again solved by a unique storage (carryover) function q_θ, which maps current supply X_t to storage quantity $q_\theta(X_t)$, and induces law of motion

$$X_{t+1} = \lambda q_\theta(X_t) + \xi_{t+1}, \quad \xi_t \sim \psi_\theta. \tag{4}$$

As before, q_θ is continuous and increasing on $[0, \infty)$. The corresponding consumption function $c_\theta(x) := x - q_\theta(x)$ is also increasing. The rational expectations pricing functional becomes $p_\theta := f \circ c_\theta$. A unique stationary distribution μ_θ exists for $(X_t)_{t=0}^\infty$ given by (4), and the stationary distribution for prices is the probability measure $\mu_\theta \circ c_\theta^{-1} \circ f^{-1}$.

It is hoped that the perturbed equilibrium $\mu_\theta \circ c_\theta^{-1} \circ f^{-1}$ converges to the true equilibrium $\mu \circ c^{-1} \circ f^{-1}$ as $\theta \to 0$. The main difficulty here is to establish that

Proposition 3 *The map $\theta \mapsto \mu_\theta$ is continuous on $\Theta := [0, \infty)$.*

Proof We cannot apply Stenflo's Proposition 1 because, among other things, $\theta \mapsto \psi_\theta$ is only continuous in the weak topology rather than total variation norm. Instead we apply Proposition 2. Here $S = Z = [0, \infty)$, both spaces being endowed with the usual metric, and $T_\theta(x, z) = \lambda q_\theta(x) + z$.

Pick any $\theta_0 \in \Theta$ and any bounded neighborhood N of θ_0. We show that Assumptions 8–11 all hold on N, in which case Proposition 2 applies, and continuity on N—and in particular at θ_0—is verified.

Regarding Assumption 8, continuity of $\theta \mapsto q_\theta(x)$ and hence $\theta \mapsto \lambda q_\theta(x) + z$ follows from standard arguments (see BBW).

Assumption 9 is also straightforward, as for any $x, x' \in S$ we have

$$\int d(T_\theta(x, z), T_\theta(x', z))\psi_\theta(dz) \leq |\lambda q_\theta(x) - \lambda q_\theta(x')| \leq |x - x'|.$$

Here the second inequality is due to the fact that $\lambda < 1$, and that both q_θ and c_θ are increasing in x.

Regarding Assumption 10, let $V(x) := x$, which is clearly a Lyapunov function on S. For any $\theta \in N$ and $x \in S$ we have

$$\begin{aligned}\int V(T_\theta(x,z))\psi_\theta(dz) &= \int T_\theta(x,z)\psi_\theta(dz) \\ &= \lambda q_\theta(x) + \int z\psi_\theta(dz) \\ &\leq \lambda x + \sup_{\theta \in N} \int z\psi_\theta(dz) = \lambda V(x) + L,\end{aligned}$$

where $L := \sup_{\theta \in N} \int z\psi_\theta(dz)$. Since N is bounded the term on the far right is finite, and Assumption 10 is verified.

Assumption 11 is also immediate, because for any $x \in S$ and $\theta \in N$ we have $d(T_\theta(x,z), T_\theta(x,z')) \leq |z - z'|$. This concludes the proof of Proposition 3.

One can verify in addition that

Proposition 4 *The perturbed equilibrium* $\mu_\theta \circ c_\theta^{-1} \circ f^{-1}$ *converges to the true equilibrium* $\mu \circ c^{-1} \circ f^{-1}$ *as* $\theta \to 0$.

The proof follows from Proposition 3 and Lemma 3 in the appendix. The details are omitted.

Example 2. A representative household maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\eta \ln c_t + (1-\eta) \ln \ell_t),$$

subject to $c_t + k_{t+1} \leq Ak_t^\alpha (1-\ell_t)^{1-\alpha} \varepsilon_{t+1}$, $\alpha \in (0,1)$. We take $(\varepsilon_t)_{t=1}^\infty$ as IID on $(0,\infty)$. It is well-known that the optimal accumulation policy for this model is given by $k_{t+1} = \alpha\beta A k_t^\alpha (1-\ell)^{1-\alpha} \varepsilon_{t+1}$, where $\ell := (1-\eta)/(1-\alpha\eta)$. Taking logs and setting $\kappa := \ln k$ and $\xi := \ln \varepsilon$ gives

$$\kappa_{t+1} = b + \alpha\kappa_t + \xi_{t+1}. \tag{5}$$

Let $\xi \sim \psi \in \mathscr{P}(\mathbb{R})$, with $\mathbb{E}|\xi| := \int |z|\psi(dz) < \infty$. Also, let $S = Z = \mathbb{R}$, and let $d(x,y) = |x - y|$. Finally, although b depends on several parameters it is sufficient for our purposes to regard it as a single parameter taking values in $\mathbb{R}$. With this convention we can take

$$\theta := (b, \alpha) \ni \mathbb{R} \times (0,1) =: \Theta,$$

and $T_\theta(\kappa, z) = b + \alpha\kappa + z$. For this model we cannot apply Stenflo's parametric continuity result, because Assumption 6 is not satisfied. To see this, take $\theta = (b, \alpha)$ and $\theta' = (b', \alpha')$ with $\alpha \neq \alpha'$. Then

$$\sup_{\kappa \in S} d(T_\theta(\kappa, z), T_{\theta'}(\kappa, z)) = \sup_{\kappa \in S} |b + \alpha\kappa + z - b' - \alpha'\kappa - z| \\ \leq |b - b'| + |\alpha - \alpha'| \sup_{\kappa \in S} |\kappa| = \infty.$$

However, Proposition 2 is easy to apply. Let N be any open subset of Θ with compact closure $\bar{N} \subset \Theta$. By Lemma 2, (5) has one and only one stationary distribution μ_θ for each $\theta \in N$, so to prove that $N \ni \theta \mapsto \mu_\theta \in \mathscr{P}(S)$ is continuous we need only verify that Assumptions 8–11 hold on N.

Assumptions 8 and 11 are completely trivial. Assumption 9 is also straightforward, because for all $\theta \in N$ we have

$$d(T_\theta(\kappa, z), T_\theta(\kappa', z)) = |b + \alpha\kappa + z - b - \alpha\kappa' - z| = \alpha|\kappa - \kappa'| \leq d(\kappa, \kappa').$$

Regarding Assumption 10, let $V(x) := |x|$, which is clearly Lyapunov on $\mathbb{R}$. Since $\bar{N}$ is a compact subset of $\Theta = \mathbb{R} \times (0, 1)$, there is a $\lambda < 1$ and an $L_0 < \infty$ such that $\alpha \leq \lambda$ and $|b| \leq L_0$ for all $(b, \alpha) \in N$. Setting $L := L_0 + \mathbb{E}|\xi|$, we get

$$\int V(T_\theta(\kappa, z))\psi(dz) = \int |b + \alpha\kappa + z|\psi(dz) \\ \leq \alpha|\kappa| + |b| + \mathbb{E}|\xi| \leq \lambda V(\kappa) + L.$$

As a result, Assumptions 8–11 are all verified, Proposition 2 applies, and $\theta \mapsto \mu_\theta$ is continuous on N.

6 Appendix

The remaining proofs are now given. We begin with some preliminary observations:

First, if $g \in bL(S, d)$ and $\|g\|_{bL} \leq r$, then $\|r^{-1}g\|_{bL} \leq 1$; from which we can see that if μ and $\mu' \in \mathscr{P}(S)$, and $g \in bL(S, d)$ with $\|g\|_{bL} \leq r$, then $|\langle \mu, g \rangle - \langle \mu', g \rangle| \leq r\varrho_{FM}(\mu, \mu')$.

Call $h \colon S \supset C \to \mathbb{R}$ K-Lipschitz on C if $|h(x) - h(x')| \leq Kd(x, x')$ for all pairs $x, x' \in C$. Below we make use of

Lemma 3 *Let $\{g_n\}_{n\in\mathbb{N}}$ be a collection of measurable real functions on S, and let $\{\mu_n\}_{n\in\mathbb{N}}\cup\{\mu\}\subset\mathscr{P}(S)$. We have $|\langle g_n,\mu_n\rangle-\langle g_n,\mu\rangle|\to 0$ as $n\to\infty$ whenever the following three conditions hold.*

1. *$\mu_n\to\mu$ in $w(\mathscr{P}(S),bC(S))$,*
2. *$\exists M<\infty$ such that $|g_n|\le M$ for all $n\in\mathbb{N}$, and*
3. *for each compact $C\subset S$, there exists a $K<\infty$ such that every g_n is K-Lipschitz on C.*

Proof Pick any $\varepsilon>0$. As S is separable and completely metrizable, any convergent sequence in $\mathscr{P}(S)$ is tight (Dudley, 2002, Theorem 11.5.3), and we can take a compact $C\subset S$ such that $\sup_n\mu_n(S\backslash C)\le\varepsilon$ and $\mu(S\backslash C)\le\varepsilon$. Moreover, the Skorohod–Dudley representation yields a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and S-valued random variables $(X_n)_{n=1}^{\infty}$ and X with X_n (resp., X) having distribution μ_n (resp., μ) and such that $X_n\to X$ as $n\to\infty$ holds $\mathbb{P}$-almost surely. Let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

Observe that

$$|\langle g_n,\mu_n\rangle-\langle g_n,\mu\rangle|=|\mathbb{E}\,g_n\circ X_n-\mathbb{E}\,g_n\circ X|\le\mathbb{E}\,|g_n\circ X_n-g_n\circ X|.$$

The far right term can be decomposed as

$$\mathbb{E}\,|g_n\circ X_n-g_n\circ X|\mathbb{1}\{X_n\in C\text{ and }X\in C\}\\ +\mathbb{E}\,|g_n\circ X_n-g_n\circ X|\mathbb{1}\{X_n\notin C\text{ or }X\notin C\}. \quad (6)$$

Since $|g_n|\le M$ and

$$|g_n\circ X_n-g_n\circ X|\mathbb{1}\{X_n\in C\text{ and }X\in C\}\le Kd(X_n,X),$$

the Dominated Convergence Theorem implies that the first term in (6) converges to zero. Regarding the second term, we have

$$|g_n\circ X_n-g_n\circ X|\mathbb{1}\{X_n\notin C\text{ or }X\notin C\}\le 2M(\mathbb{1}\{X_n\notin C\}+\mathbb{1}\{X\notin C\}).$$

$$\therefore\quad 0\le\limsup_{n\to\infty}|\langle g_n,\mu_n\rangle-\langle g_n,\mu\rangle|\le 4M\varepsilon.$$

As ε is arbitrary the claim is proved.

Proof (Proof of Proposition 1) First we verify Assumption 1. To do so, pick any (θ,μ) in $N\times\mathscr{P}(S)$, and any sequence $(\theta_n,\mu_n)_{n=1}^{\infty}\subset N\times\mathscr{P}(S)$ converging to (θ,μ). Let $h\in bL(S,d)$, $\|h\|_{bL}\le 1$. We need to show that

$$|\langle\mu_nP_{\theta_n},h\rangle-\langle\mu P_\theta,h\rangle|=|\langle P_{\theta_n}h,\mu_n\rangle-\langle P_\theta h,\mu\rangle|\to 0\quad(n\to\infty). \quad (7)$$

This will be true if each of the terms in the dominating sum

$$|\langle P_{\theta_n} h, \mu_n\rangle - \langle P_{\theta_n} h, \mu\rangle| + |\langle P_{\theta_n} h, \mu\rangle - \langle P_\theta h, \mu\rangle| \tag{8}$$

converges to zero.

Claim 1. Under the hypotheses of the proposition, the first term in (8) converges to zero as $n \to \infty$.

Define $g_n(x) := P_{\theta_n} h(x)$. Evidently $|g_n| \leq |h| \leq 1$, and

$$\begin{aligned}
|g_n(x) - g_n(x')| &= \left| \int h(T_{\theta_n}(x,z))\psi_{\theta_n}(dz) - \int h(T_{\theta_n}(x',z))\psi_{\theta_n}(dz) \right| \\
&\leq \int |h(T_{\theta_n}(x,z)) - h(T_{\theta_n}(x',z))| \psi_{\theta_n}(dz) \\
&\leq \int d(T_{\theta_n}(x,z), T_{\theta_n}(x',z))\psi_{\theta_n}(dz).
\end{aligned}$$

Assumption 4 now gives

$$|g_n(x) - g_n(x')| \leq \lambda d(x,x'), \quad \forall x, x' \in S, \;\; \forall n \in \mathbb{N}. \tag{9}$$

It follows that $g_n \in bL(S,d)$ and $\|g_n\|_{bL} \leq 2$ for all n; and hence

$$|\langle P_{\theta_n} h, \mu_n\rangle - \langle P_{\theta_n} h, \mu\rangle| = |\langle g_n, \mu_n\rangle - \langle g_n, \mu\rangle| \leq 2\varrho_{FM}(\mu_n, \mu) \to 0.$$

Claim 2. Under the hypotheses of the proposition, the second term in (8) converges to zero as $n \to \infty$.

Clearly

$$\begin{aligned}
&|\langle P_{\theta_n} h, \mu\rangle - \langle P_\theta h, \mu\rangle| \\
&\qquad \leq \int \left| \int h(T_{\theta_n}(x,z))\psi_{\theta_n}(dz) - \int h(T_\theta(x,z))\psi_\theta(dz) \right| \mu(dx).
\end{aligned}$$

Fix $x \in S$ and consider the term inside the absolute value symbols. It is dominated by

$$\begin{aligned}
&\left| \int h(T_{\theta_n}(x,z))\psi_{\theta_n}(dz) - \int h(T_\theta(x,z))\psi_{\theta_n}(dz) \right| \\
&\qquad + \left| \int h(T_\theta(x,z))\psi_{\theta_n}(dz) - \int h(T_\theta(x,z))\psi_\theta(dz) \right|.
\end{aligned} \tag{10}$$

From Assumption 6, the first term in this sum is bounded above by

$$\begin{aligned}
&\int |h(T_{\theta_n}(x,z)) - h(T_\theta(x,z))| \psi_{\theta_n}(dz) \\
&\qquad \leq \int d(T_{\theta_n}(x,z), T_\theta(x,z))\psi_{\theta_n}(dz) \leq \delta(e(\theta_n, \theta)).
\end{aligned} \tag{11}$$

Since $|h| \leq 1$, the second term in the sum (10) is bounded above by $\|\psi_{\theta_n} - \psi_\theta\|$, where $\|\cdot\|$ is the total variation norm on $\mathscr{P}(Z)$.

Since $x \in S$ was arbitrary and μ is a probability measure we have

$$|\langle P_{\theta_n} h, \mu\rangle - \langle P_\theta h, \mu\rangle| \leq \delta(e(\theta_n, \theta)) + \|\psi_{\theta_n} - \psi_\theta\|. \tag{12}$$

Claim 2 now follows from Assumptions 6 and 7. The required continuity of $(\theta, \mu) \mapsto \mu P_\theta$ is verified.

Next we prove Assumptions 2 and 3 with $V(x) = W(x) = d(x, x_0)$ and $M = L/(1-\lambda)$. Bounded compactness of S implies that $V \in \mathscr{L}(S)$. For fixed $x \in S$ we have

$$\begin{aligned} P_\theta V(x) &= \int V(T_\theta(x, z))\psi_\theta(dz) \\ &= \int d(T_\theta(x, z), x_0)\psi_\theta(dz) \\ &\leq \int d(T_\theta(x, z), T_\theta(x_0, z))\psi_\theta(dz) + \int d(T_\theta(x_0, z), x_0)\psi_\theta(dz) \\ &\leq \lambda V(x) + L. \end{aligned}$$

Since x was arbitrary, we have

$$P_\theta V \leq \lambda V + L \quad \text{pointwise on S.}$$

Iterating on this inequality, and using the fact that P_θ is positive and linear with $P_\theta \mathbb{1}_S = \mathbb{1}_S$, we get

$$P_\theta^t V \leq \lambda^t V + \lambda^{t-1} L + \lambda^{t-2} L + \cdots + L.$$

This and the fact that λ and L are independent of θ with $\lambda < 1$ provides the uniform bound

$$\sup_{\theta \in N} \sup_{t \geq 1} P_\theta^t V \leq V + \frac{L}{1-\lambda}.$$

In particular, for $x = x_0$ we get $\sup_{\theta \in N} \sup_{t \geq 1} P_\theta^t V(x_0) \leq L/(1-\lambda)$, which verifies Assumption 2.

Now let $V_n := V \wedge n$ be the n-th truncation of V, and let μ_θ be the stationary distribution corresponding to θ. Since $V_n \in bC(S)$, $\forall n \in \mathbb{N}$, Lemma 2 and the definition of convergence in $w(\mathscr{P}(S), bC(S))$ imply that

$$\lim_t P_\theta^t V_n(x_0) = \int V_n d\mu_\theta, \quad \forall n \in \mathbb{N}. \tag{13}$$

Also, since P_θ and hence P_θ^t are positive operators, we have $P_\theta^t V_n(x_0) \leq P_\theta^t V(x_0)$, which in turn is bounded by $L/(1-\lambda)$. The Monotone Convergence Theorem now gives

$$\int V d\mu_\theta = \lim_n \int V_n d\mu_\theta = \lim_n \lim_t P_\theta^t V_n(x_0) \leq \frac{L}{1-\lambda}, \quad \forall \theta \in N.$$

Assumption 3 is therefore satisfied with $W(x) = V(x) = d(x, x_0)$ and $M = L/(1-\lambda)$.

Proof (Proof of Proposition 2) We verify below that Assumptions 1–3 hold. Let's start with Assumption 1. As in the proof of Proposition 1, let $(\theta, \mu) \in N \times \mathscr{P}(S)$, and let $(\theta_n, \mu_n)_{n=1}^\infty \subset N \times \mathscr{P}(S)$ be a sequence converging to (θ, μ). Fix $h \in bL(S, d)$, $\|h\|_{bL} \leq 1$. As in the proof of Assumption 1 in Proposition 1, we proceed by establishing that both terms in (8) converge to zero.

Claim 1. Under the hypotheses of the proposition, the first term in (8) converges to zero as $n \to \infty$.

To prove Claim 1, define again $g_n(x) := P_{\theta_n} h(x)$. Under this notation, we are seeking to establish that

$$|\langle g_n, \mu_n \rangle - \langle g_n, \mu \rangle| \to 0 \quad (n \to \infty).$$

This will hold if the conditions of Lemma 3 are established. Since $\mu_n \to \mu$ by hypothesis, we need only check Conditions 2 and 3 of the lemma. Evidently $|g_n| \leq |h| \leq 1$ Moreover, by $\|h\|_{bL} \leq 1$,

$$\begin{aligned}
|g_n(x) - g_n(x')| &= \left| \int h(T_{\theta_n}(x, z)) \psi_{\theta_n}(dz) - \int h(T_{\theta_n}(x', z)) \psi_{\theta_n}(dz) \right| \\
&\leq \int |h(T_{\theta_n}(x, z)) - h(T_{\theta_n}(x', z))| \psi_{\theta_n}(dz) \\
&\leq \int d(T_{\theta_n}(x, z), T_{\theta_n}(x', z)) \psi_{\theta_n}(dz).
\end{aligned}$$

Pick any compact $C \subset S$. Assumption 9 gives

$$|g_n(x) - g_n(x')| \leq K d(x, x'), \quad \forall x, x' \in C, \ \forall n. \tag{14}$$

Condition 3 of Lemma 3 then holds, and the claim is proved.

Claim 2. Under the hypotheses of the proposition, the second term in (8) converges to zero as $n \to \infty$.

To prove the claim, note that $|\langle P_{\theta_n} h, \mu\rangle - \langle P_\theta h, \mu\rangle|$ is equal to

$$\left| \int \int h(T_{\theta_n}(x,z))\psi_{\theta_n}(dz)\mu(dx) - \int \int h(T_\theta(x,z))\psi_\theta(dz)\mu(dx) \right|$$
$$\leq \int \left| \int h(T_{\theta_n}(x,z))\psi_{\theta_n}(dz) - \int h(T_\theta(x,z))\psi_\theta(dz) \right| \mu(dx).$$

If we fix $x \in S$ and define

$$r_n(z) := h(T_{\theta_n}(x,z)), \quad r(z) = h(T_\theta(x,z)),$$

then by the Dominated Convergence Theorem it is sufficient for Claim 2 to show that

$$|\langle r_n, \psi_n\rangle - \langle r, \psi\rangle| \to 0 \quad (n \to \infty),$$

where we are writing ψ_n for ψ_{θ_n} and ψ for ψ_θ. But

$$|\langle r_n, \psi_n\rangle - \langle r, \psi\rangle| \leq |\langle r_n, \psi_n\rangle - \langle r_n, \psi\rangle| + |\langle r_n, \psi\rangle - \langle r, \psi\rangle|.$$

Moreover, that $|\langle r_n, \psi\rangle - \langle r, \psi\rangle| \to 0$ is immediate from Assumption 8 and the Dominated Convergence Theorem. Hence it remains only to show that

$$|\langle r_n, \psi_n\rangle - \langle r_n, \psi\rangle| \to 0 \quad (n \to \infty). \tag{15}$$

That this holds true under the total variation convergence of ψ_n to ψ in the first condition of Assumption 11 is immediate. Let us now establish the same under the alternative hypothesis in Assumption 11.

We check the conditions of Lemma 3. That $\psi_n \to \psi$ in $w(\mathscr{P}(S), bC(S))$ is true by hypothesis. Evidently $|r_n| \leq |h| \leq 1$, so Condition 2 of the lemma holds. Moreover, by $\|h\|_{bL} \leq 1$,

$$|r_n(z) - r_n(z')| \leq d(T_{\theta_n}(x,z), T_{\theta_n}(x,z')).$$

Fix $C \subset S$, compact. Assumption 11 gives

$$|r_n(z) - r_n(z')| \leq J d(z,z'), \quad \forall z, z' \in C, \ \ \forall n. \tag{16}$$

Condition 3 of the lemma therefore holds, and the claim is verified.

Next we check Assumption 2. An identical argument to the iterative procedure used in the proof of Proposition 1 yields

$$\sup_{\theta \in N} \sup_{t \geq 1} P_\theta^t V(x) \leq V(x) + \frac{L}{1-\lambda}. \tag{17}$$

That Assumption 2 holds is immediate from this bound.

Since Assumptions 1 and 2 hold the first part of the proposition is verified: $\Lambda(\theta)$ is nonempty for each $\theta \in N$. To prove the second assertion we need only check Assumption 3 under the hypothesis that $\Lambda(\theta) = \{\mu_\theta\}$ is single-valued. Define from P_θ the new operator $\bar{P}_\theta$ by $\bar{P}_\theta := t^{-1} \sum_{j=1}^{t} P_\theta^j$. By Meyn and Tweedie (1993, Proposition 12.1.4), $\delta_x \bar{P}^t \to \mu_\theta$ as $t \to \infty$ for all $x \in S$. Repeating exactly the verification of Assumption 3 in Proposition 1, but replacing P_θ by $\bar{P}_\theta$, we can see that Assumption 3 also holds under the hypotheses of Proposition 2. The proof is done.

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