Stochastic Optimal Growth with Nonconvexities ⋆

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Abstract

This paper studies optimal investment and dynamic behavior in stochastically growing economies. We assume neither convex technology nor bounded support of the productivity shocks. A number of basic results concerning the investment policy and the Ramsey–Euler equation are established. We also prove a fundamental dichotomy pertaining to optimal growth models perturbed by standard econometric shocks: Either an economy is globally stable or it is globally collapsing to the origin.

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1 Introduction

The stochastic optimal growth model (Brock and Mirman, 1972) is a foundation stone of modern macroeconomic and econometric research. To accommodate the data, however, economists are often forced to go beyond the convex production technology used in these original studies. Nonconvexities lead to technical difficulties which applied researchers would rather not confront. Value functions are in general no longer smooth, optimal policies contain jumps, and the Euler equation may not hold. This reality precludes the use of many standard tools. Further, convergence of state variables to a stationary equilibrium is no longer assured. The latter is a starting point of much applied analysis (see, e.g., Kydland and Prescott, 1982; or Long and Plosser, 1983) and fundamental to the rational expectations hypothesis (Lucas, 1986).

Although nonconvexities are technically challenging, the richer dynamics that they provide help to replicate key time series. For example, nonconvexities often lead to the kind of regime-switching behavior found in aggregate income data (e.g., Prescott, 2003), or the growth miracles and growth disasters in cross-country income panels. Also, nonconvexities can arise directly from micro-level modeling, taking the form of fixed costs, threshold effects, ecological properties of natural resource systems, economies of scale and scope, network and agglomeration effects, and so on.

The objective of this paper is to investigate in depth the fundamental properties of stochastic nonconvex one-sector models and the series they generate using assumptions which facilitate integration with empirical research. [1]

Previously, in the deterministic case, optimal growth models with nonconvex technology were studied in continuous time by Skiba (1978). In discrete time, Majumdar and Mitra (1982) examined efficiency of intertemporal allocations. Dechert and Nishimura (1983) studied the standard discounted model with convex/concave technology, and characterized the dynamics of the model for every value of the discount factor. More recently, Amir, Mirman and Perkins (1991) used lattice programming techniques to study solutions of the Bellman equation and associated comparative dynamics. Kamihigashi and Roy (2005) study nonconvex optimal growth without differentiability or even continuity.

In the stochastic case, a rigorous early treatment of optimal growth with nonconvex technology is given in Majumdar, Mitra and Nyarko (1989). Amir (1997) studies optimal growth in economies that have some degree of convexity. Using martingale arguments, Joshi (1997) analyzes the classical turnpike properties when technology is nonstationary. Schenk-Hoppé (2005) considers dynamic stability of stochastic overlapping generations models with S-shaped

[1] We consider only optimal dynamics. There are many studies of nonoptimal competitive dynamics in nonconvex environments. See for example Mirman, Morand and Reffett (2005).

production function. Mitra and Roy (2005) study nonconvex renewable resource exploitation and stability of the resource stock.

The above papers assume that the shock which perturbs activity in each period has compact support. We extend their analysis by assuming instead that the shock is multiplicative, and its distribution has a density—which may in general have bounded or unbounded support. This formulation is relatively standard in quantitative applications. It provides considerable structure, which can be exploited when investigating optimality and dynamics.

Without convexity many standard results pertaining to the optimal policy and the value function can fail. In this study the density representation of the shock is used to prove interiority of the optimal policy and Ramsey–Euler type results. Based on these findings and some additional assumptions, we then obtain a fundamental dichotomy: Every economy is either globally stable in a strong sense to be made precise, or globally collapsing to the origin. This result (a version of the Foguel Alternative) simplifies greatly the range of possible dynamics. We connect the two possibilities to the discount rate, and also provide conditions to determine which outcome prevails for specific parameterizations. [2]

One caveat is that we consider only one-sector models. Multi-sector models are common in applications, but their dynamics are vastly more complex. Thus it is an important open question whether or not the results on dynamics presented in this paper can be extended to multi-sector models.

Section 2 introduces the model. Section 3 discusses optimization and properties of the optimal policies. Section 4 considers the dynamics of the processes generated by these policies (i.e., the optimal paths). All of the proofs are given in Section 5 and the appendix.

2 Outline of the Model

Let $\mathbb{R}_+ := [0, \infty)$ and let $\mathscr{B}$ be the Borel subsets of $\mathbb{R}_+$. At the start of each period t a representative agent receives current income $y_t \in \mathbb{R}_+$ and allocates it between current consumption c_t and savings. On current consumption c the agent receives instantaneous utility $u(c)$. Savings determines the stock k_t of available capital, where $0 \leq k_t + c_t \leq y_t$. Production then takes place, delivering at the start of the next period output

$$y_{t+1} = f(k_t)\varepsilon_t, \tag{1}$$

which is net of depreciation. Here ε_t is a shock taking values in $\mathbb{R}_+$.

[2] For further discussion of dynamics, including a specific condition on the primitives that ensures global stability, see Nishimura and Stachurski (2005).

The productivity shocks $(\varepsilon_t)_{t=0}^{\infty}$ form an independent and identically distributed sequence on probability space $(\Omega, \mathscr{F}, \mathbf{P})$. When the time t savings decision is made shocks $\varepsilon_0, \ldots, \varepsilon_{t-1}$ are observable. The distribution of ε_t is represented by density φ.[3] We let $\mathbb{E}[\varepsilon_t] = \int z\varphi(z)dz = 1$. The bold symbol $\boldsymbol{\varphi}$ is used to denote the probability measure on $\mathbb{R}_+$ corresponding to the density φ, so that $\boldsymbol{\varphi}(dz)$ and $\varphi(z)dz$ have the same meaning.

The agent seeks to maximize the expectation of a discounted sum of utilities. Future utility is discounted according to $\varrho \in (0, 1)$.

Assumption 2.1 *The function u is strictly increasing, strictly concave, and continuously differentiable on* $(0, \infty)$. *It satisfies* (U1) $\lim_{c \to 0} u'(c) = \infty$; *and* (U2) *$u$ is bounded with $u(0) = 0$.*

The condition (U1) is needed to obtain the Ramsey–Euler equation. Strict concavity is critical to the proof of monotonicity of the optimal policy, on which all subsequent results depend. Note that if u is required to be bounded, then assuming $u(0) = 0$ sacrifices no additional generality.

Assumption 2.2 *The production function f is strictly increasing and continuously differentiable on* $(0, \infty)$. *In addition,* (F1) $f(0) = 0$; (F2) $\lim_{k \to \infty} f'(k) = 0$; *and* (F3) $\lim_{k \to 0} f'(k) > 1$.

Condition (F2) is the usual decreasing returns assumption. Actually for the proofs we require only that f is majorized by an affine function with slope less than one. This is implied by (F2), as can be readily verified from the Fundamental Theorem of Calculus.

An economy is defined by the collection (u, f, φ, ϱ), for which Assumptions 2.1 and 2.2 are always taken to hold.

By a control policy is meant a function $\pi \colon \mathbb{R}_+ \ni y \mapsto k \in \mathbb{R}_+$ associating current income to current savings. The policy is called feasible if it is $\mathscr{B}$-measurable and $0 \leq \pi(y) \leq y$ for all y. An initial condition and a feasible policy complete the dynamics of the model (1), determining a stochastic process $(y_t)_{t \geq 0}$ on $(\Omega, \mathscr{F}, \mathbf{P})$, where $y_{t+1} = f(\pi(y_t))\varepsilon_t$ for all $t \geq 0$.

Investment behavior is determined by the solution to the problem

$$\sup_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(y_t - \pi(y_t))\right], \tag{2}$$

where $\mathbb{E}$ denotes integration over Ω with respect to $\mathbf{P}$, an initial condition y_0 is given, and the supremum is over the set of all feasible policies. By (U2) the functional inside the integral is bounded independent of π, and the supremum always exists. A policy is called optimal if it is feasible and attains the supremum (2).

[3] That is, $\mathbf{P}[\varepsilon_t^{-1}(B)] = \int_B \varphi(z)dz$ for all $B \in \mathscr{B}$. Here and in what follows, by a density is meant a nonnegative $\mathscr{B}$-measurable function on $\mathbb{R}_+$ that integrates to unity.

3 Optimization

In this section we solve the optimization problem by dynamic programming, and characterize the properties of the value function and control policy. To begin, define as usual the value function V by setting $V(y)$ as the real number given in (2) when $y = y_0$ is the initial condition. Let $b\mathscr{B}$ be the space of real bounded $\mathscr{B}$-measurable functions on $\mathbb{R}_+$. Define also the Bellman operator T mapping $b\mathscr{B}$ into itself by

$$(Tv)(y) = \sup_{0 \leq k \leq y} \left\{ u(y-k) + \varrho \int v[f(k)z]\boldsymbol{\varphi}(dz) \right\}. \tag{3}$$

It is well-known that T is a uniform contraction on $b\mathscr{B}$ in the sense of Banach, and that the value function V is the unique fixed point of T in $b\mathscr{B}$.

Lemma 3.1 *For any economy (u, f, φ, ϱ), the value function V is continuous, bounded and strictly increasing. An optimal policy π exists. Moreover, if π is optimal, then*

$$V(y) = u(y - \pi(y)) + \varrho \int V[f(\pi(y))z]\boldsymbol{\varphi}(dz), \quad \forall y \in \mathbb{R}_+.$$

The proof does not differ from the neoclassical case (see for example Stokey et al., 1989) and is omitted.

As a matter of notation, define

$$\Sigma(y) := \mathrm{argmax}_{0 \leq k \leq y} \left\{ u(y-k) + \varrho \int V[f(k)z]\boldsymbol{\varphi}(dz) \right\},$$

so that $y \mapsto \Sigma(y)$ is the optimal correspondence, and π is an optimal policy if and only if it is a $\mathscr{B}$-measurable selection from Σ.

3.1 Monotonicity

Monotone policy rules play an important role in economics, particularly with regards to the characterization of equilibria. That monotonicity of the optimal investment function holds in deterministic one-sector nonconvex growth environments was established by Dechert and Nishimura (1983) and is now well-known. Indeed, monotone controls are a feature of many very general environments. See for example Mirman, Morand and Reffett (2005) and Kamihigashi and Roy (2005). Our model is no exception:

Lemma 3.2 *Let an economy (u, f, φ, ϱ) be given, and let π be a feasible policy. If π is optimal, then it is nondecreasing on $\mathbb{R}_+$.*

Put differently, one cannot construct a measurable selection from the optimal correspondence Σ that is not nondecreasing. (On the other hand, in nonconvex models consumption is *not* generally monotone with income.)

One supposes that as ϱ decreases the propensity to save will fall. The following result was established for the stochastic neoclassical case in Danthine and Donaldson (1981, Theorem 5.1), and in the nonconvex, deterministic case by Amir, Mirman and Perkins (1991) using lattice programming.

Lemma 3.3 *The optimal policy is nondecreasing in the discount factor ϱ, in the sense that if $(u, f, \varphi, \varrho_0)$ and $(u, f, \varphi, \varrho_1)$ are two economies, and if π_0 (resp. π_1) is optimal for the former (resp. latter), then $\varrho_1 \geq \varrho_0$ implies $\pi_1 \geq \pi_0$ pointwise on $\mathbb{R}_+$.*

In fact we can say more:

Lemma 3.4 *For u, f and φ given, let (ϱ_n) be a sequence of discount factors in $(0, 1)$, and for each n let π_n be a corresponding optimal policy. If $\varrho_n \downarrow 0$, then $\pi_n \downarrow 0$ pointwise, and the convergence is uniform on compact sets.*

3.2 Derivative characterization of the policy

Optimal behavior in growth models is usually characterized by the Ramsey–Euler equation. In stochastic models, where sequential arguments are unavailable, the obvious path to the Ramsey–Euler equation is via differentiability of the value function and a well-known envelope condition (Mirman and Zilcha, 1975, Lemma 1). See also Blume, Easley and O'Hara (1982), who demonstrated differentiability of the optimal *policy* under convexity and absolute continuity of the shock by way of the Implicit Function Theorem. Amir (1997) considered a weaker convexity requirement.

Without any convexity there may be jumps in the optimal policy, which in turn affect the smoothness of the value function. The validity of the Ramsey–Euler characterization is not clear. However, Dechert and Nishimura (1983, Theorem 6, Lemma 8) showed that in their model the value function has left and right derivatives at every point, and that these agree off an at most countable set. [4] These results were extended to the stochastic case by Majumdar, Mitra and Nyarko (1989). In addition to the above results concerning the value function, they were able to show that the Ramsey–Euler equations holds *everywhere*, irrespective of jumps in the optimal policy. We extend their analysis, starting from the essential idea of Blume, Easley and O'Hara (1982), but without

[4] The intuition is that nondifferentiability of the value function coincides pointwise with jumps in optimal investment. But by Lemma 3.2, the only optimal jumps are increases. To each jump, then, can be associated a distinct rational, which precludes uncountability.

convexity or compact state. From this we prove interiority of the policy and the Ramsey–Euler equation for standard econometric shocks.

Assumption 3.1 *The shock ε_t is such that* (S1) *the density φ is continuously differentiable on* $(0, \infty)$*; and* (S2) *the integral $\int z|\varphi'(z)|dz$ is finite.*

The set of densities satisfying (S1) and (S2) is norm-dense in the set of all densities when the latter are viewed as a subset of $L_1(\mathbb{R}_+)$. They also hold for many standard econometric shocks on $\mathbb{R}_+$, such as the lognormal distribution. With these assumptions in hand we can establish the following without convexity or bounded shocks.

Proposition 3.1 *Let (u, f, φ, ϱ) satisfy Assumptions 2.1–3.1.*

1. *If policy π is optimal, then it is interior. That is, $0 < \pi(y) < y$ for all $y \in (0, \infty)$.*
2. *The value function V has right and left derivatives V'_- and V'_+ everywhere on $(0, \infty)$.*
3. *If policy π is optimal, then it satisfies $V'_-(y) \leq u'(y - \pi(y)) \leq V'_+(y)$ for all $y \in (0, \infty)$.*
4. *The functions V'_- and V'_+ disagree on an at most countable subset of $\mathbb{R}_+$.*

In the stochastic nonconvex case, Part 1 of Proposition 3.1 was proved by Majumdar, Mitra and Nyarko (1989, Theorem 4). Their proof requires that the shock has compact support bounded away from zero, and there exists an $a > 0$ such that $f(k)\varepsilon > k$ with probability one whenever $k \in (0, a)$. Part 2 was proved in the deterministic case by Dechert and Nishimura, as was Part 4 (1983, Theorem 6 and Lemma 8).[5] Part 3 is due in the stochastic neoclassical case to Mirman and Zilcha (1975, Lemma 1), and the proof for the nonconvex case is the same.[6]

Corollary 3.1 *For a given economy (u, f, φ, ϱ), any two optimal policies are equal almost everywhere.*

PROOF. Immediate from Parts 3 and 4.

It will turn out that under the maintained assumptions, differences on null sets do not really concern us (see Lemma 4.1). So we can in some sense talk about *the* optimal policy (when a.e.-equivalent policies are identified).

One of our main results is that under Assumptions 2.1–3.1 the Ramsey–Euler equation can still be established.

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[5] On Part 2 see also Askri and Le Van (1998, Proposition 3.2) and Mirman, Morand and Reffett (2005).

[6] Note that if V is concave on some open interval, then the subdifferentials exist everywhere on that interval, and $V'_+ \leq V'_-$. If follows from Part 3 of the Proposition, then, that concavity immediately gives differentiability, and, moreover, $V'(y) = u'(y - \pi(y))$. See Mirman and Zilcha (1975, Lemma 1).

Proposition 3.2 *Let Assumptions 2.1–3.1 hold. If π is optimal for (u, f, φ, ϱ), then for all $y > 0$,*

$$u'(y - \pi(y)) = \varrho \int u'[f(\pi(y))z - \pi(f(\pi(y))z)]f'(\pi(y))z\boldsymbol{\varphi}(dz).$$

Using Proposition 3.2 we can strengthen the monotonicity result for the optimal policy (Lemma 3.2). The proof is straightforward and is omitted.

Corollary 3.2 *For a given economy (u, f, φ, ϱ), every optimal policy is strictly increasing.*

4 Dynamics

Next we discuss the dynamics of the optimal process $(y_t)_{t\geq 0}$. For the nonconvex deterministic case a detailed characterization of dynamics was given by Dechert and Nishimura (1983). Not surprisingly, for some parameter values multiple equilibria obtain. On the other hand, for the convex stochastic growth model, Mirman (1970) and Brock and Mirman (1972) proved that the sequence of marginal distributions for the process converge to a unique limit independent of the initial condition. Subsequently this problem has been treated by many authors. [7]

Our main contribution in this paper is to show that many convex and nonconvex optimal processes satisfy a fundamental dichotomy: Either they are globally stable, or they are globally collapsing to the origin, independent of the initial condition. This result reduces considerably the possible range of asymptotic outcomes. For example, path dependence never holds. More importantly, global stability can now be established by showing only that an economy does not collapse to the origin.

To begin, let $\mathscr{P}$ be the set of probability measures on $(\mathbb{R}_+, \mathscr{B})$. Let $E := (u, f, \varphi, \varrho)$ be given. For a fixed policy π and initial condition y_0, we consider the evolution of the income process $(y_t)_{t\geq 0}$ satisfying $y_{t+1} = f(\pi(y_t))\varepsilon_t$, and the corresponding sequence of marginal distributions $(\boldsymbol{\psi}_t)_{t\geq 0} \subset \mathscr{P}$. [8] Evidently the process is Markovian, with y_t independent of ε_t. From this independence it follows that for any bounded Borel function $h \colon \mathbb{R}_+ \to \mathbb{R}$ we have

$$\mathbb{E}h(y_{t+1}) = \mathbb{E}h[f(\pi(y_t))\varepsilon_t] = \int \int h[f(\pi(y))z]\boldsymbol{\varphi}(dz)\boldsymbol{\psi}_t(dy).$$

[7] See Nishimura and Stachurski (2005) and references.

[8] As before, $(y_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathscr{F}, \mathbf{P})$. By the marginal distribution $\boldsymbol{\psi}_t \in \mathscr{P}$ of y_t is meant its distribution on $\mathbb{R}_+$ in the usual sense. Precisely, $\boldsymbol{\psi}_t := \mathbf{P} \circ y_t^{-1}$, the image measure induced on $(\mathbb{R}_+, \mathscr{B})$ by y_t.

Specializing to the case $h = \mathbb{1}_B$ and using $y_t \sim \boldsymbol{\psi}_t$ gives the recursion

$$\boldsymbol{\psi}_{t+1}(B) = \int \left[\int \mathbb{1}_B[f(\pi(y))z] \boldsymbol{\varphi}(dz) \right] \boldsymbol{\psi}_t(dy). \tag{4}$$

When π is optimal for E, the sequence of marginal distributions $(\boldsymbol{\psi}_t)$ defined inductively by (4) is called an optimal path for (E, π). Every initial condition $y_0 \sim \boldsymbol{\psi}_0$ defines a (unique) optimal path. If initial income is zero the dynamics require no additional investigation. Henceforth, by an *initial condition* is meant a distribution $\boldsymbol{\psi_0} \in \mathscr{P}$ for y_0 which puts no mass on $\{0\}$. This convention makes the results neater, and is maintained throughout the proofs without further comment.

When studying convergence of probabilities two topologies are commonly used. One is the so-called weak topology, under which distribution functions converge if and only if they converge pointwise at all continuity points. The other is the norm topology, or strong topology, generated by the total variation norm. Under the latter, the distance between μ and ν in $\mathscr{P}$ is $\sup_{B \in \mathscr{B}} |\mu(B) - \nu(B)|$.

Definition 4.1 *Let an economy $E := (u, f, \varphi, \varrho)$ be given, and let π be an optimal policy for E. A (nontrivial, stochastic) steady state for (E, π) is a measure $\boldsymbol{\psi}^* \in \mathscr{P}$, such that $\boldsymbol{\psi}^*(\{0\}) = 0$ and*

$$\int \left[\int \mathbb{1}_B[f(\pi(y))z] \boldsymbol{\varphi}(dz) \right] \boldsymbol{\psi}^*(dy) = \boldsymbol{\psi}^*(B), \quad \forall B \in \mathscr{B}. \tag{5}$$

The policy π is called globally stable if for (E, π) there is a unique steady state $\boldsymbol{\psi}^* \in \mathscr{P}$, and the (E, π)-optimal path $(\boldsymbol{\psi}_t)$ satisfies $\boldsymbol{\psi}_t \to \boldsymbol{\psi}^*$ in the norm topology as $t \to \infty$ for all initial conditions $\boldsymbol{\psi}_0$.

It is clear from (4) and (5) that if $y_t \sim \boldsymbol{\psi}^*$, then $y_{t+k} \sim \boldsymbol{\psi}^*$ for all $k \geq 0$. Note also that the stability condition in Definition 4.1 is particularly strong. It implies many standard stability conditions for Markov processes, such as recurrence, and also convergence of the marginal distributions in the weak topology. [9]

Instability of stochastic growth models has been studied less than stability. There are various notions which capture instability; we borrow a relatively strong one from the Markov process literature referred to as sweeping. [10]

Definition 4.2 *Let an economy $E := (u, f, \varphi, \varrho)$ be given, and let π be an optimal policy. Let $\mathscr{B}_0 \subset \mathscr{B}$. In general, the Markov process generated by (E, π) is called sweeping with respect to $\mathscr{B}_0$ if each optimal path $(\boldsymbol{\psi}_t)$ satisfies $\lim_{t \to \infty} \boldsymbol{\psi}_t(A) = 0$ for all $A \in \mathscr{B}_0$ and all initial conditions $\boldsymbol{\psi}_0$. We say that*

[9] In the present case it also implies uniform convergence of distribution functions, which is the criterion of Brock and Mirman (1972). See Dudley (2002, p. 389).
[10] See, for example, Lasota and Mackey (1994, Section 5.9).

(E, π) is globally collapsing to the origin if it is sweeping with respect to the collection of intervals $\mathscr{B}_0 := \{[a, \infty) : a > 0\}$.

Nonconvex technology introduces the possibility that many optimal policies exist for the one economy. For these models it has been shown (Dechert and Nishimura, 1983, Lemma 6) that different optimal trajectories can have very different dynamics, even from the same initial condition. Indeed, there may be two optimal policies π and π' for E such that the optimal path from $\boldsymbol{\psi}_0$ generated by (E, π) sustains a nontrivial long run equilibrium, whereas that generated by (E, π') leads to economic collapse. For our stochastic model this is not possible:

Lemma 4.1 *Let an economy $E := (u, f, \varphi, \varrho)$ be given. If (E, π) is globally stable for some optimal π, then (E, π') is globally stable for every optimal policy π'. Similarly, if (E, π) is globally collapsing to the origin, then so is (E, π') for every optimal policy π'.*

As a result we may simply say that E is globally stable or globally collapsing, without specifying the particular optimal policy π. Next, we introduce a new assumption as a preliminary to our main dynamics result.

Assumption 4.1 *The density φ of the productivity shock is strictly positive (Lebesgue almost) everywhere on $\mathbb{R}_+$.*

Many standard shocks on $\mathbb{R}_+$ have this property. An example is the lognormal distribution. The following result indicates that when this assumption holds there is a fundamental dichotomy for the dynamic behavior of the economy. The proof is based on the Foguel Alternative for Markov chains (Foguel 1969, Rudnicki 1995). Monotonicity and interiority of the optimal policy also play key roles.

Proposition 4.1 *Let an economy $E := (u, f, \varphi, \varrho)$ be given. If in addition to Assumptions 2.1–3.1, Assumption 4.1 also holds, then there are only two possibilities. Either*

1. *E is globally stable, or*
2. *E is globally collapsing to the origin.*

Remark. Assumption 4.1 can be weakened at the cost of more complicated proofs. See Rudnicki (1995, Lemma 3 and Theorem 2).

It follows that multiple long run equilibria are never observed, regardless of nonconvexities in production technology. Instead long run outcomes are completely determined by the structure of the model, and historical conditions are asymptotically irrelevant. However, the steady state distribution may well be multi-modal, concentrated on areas that are locally attracting on average.

We have seen that a decrease in ϱ is associated with lower savings and investment, which in turn should increase the likelihood of collapse to the origin. Conversely, higher ϱ should increase the likelihood that the economy is stable. Indeed,

Lemma 4.2 *For economies $E_0 := (u, f, \varphi, \varrho_0)$ and $E_1 := (u, f, \varphi, \varrho_1)$ with $\varrho_0 \leq \varrho_1$, the following implications hold.*

1. *If E_1 is globally collapsing to the origin, then so is E_0.*
2. *If E_0 is globally asymptotically stable, then so is E_1.*

Combining the above results we can deduce that the dynamic behavior of the stochastic optimal growth model has only three possible types. Precisely,

Proposition 4.2 *For u, f, and φ given, either*

1. *(u, f, φ, ϱ) is globally stable for all $\varrho \in (0, 1)$,*
2. *(u, f, φ, ϱ) is globally collapsing for all $\varrho \in (0, 1)$, or*
3. *there is a $\hat{\varrho} \in (0, 1)$ such that (u, f, φ, ϱ) is globally stable for all $\varrho > \hat{\varrho}$, and globally collapsing for all $\varrho < \hat{\varrho}$.*

Under the current hypotheses one cannot rule out either of the first two possibilities. For example, Kamihigashi (2003) shows that very general one-sector growth models converge almost surely to zero when $f'(0) < \infty$ and shocks are sufficiently volatile. Determining which of the above three possibilities holds is far from trivial. However, we now show that one need only consider behavior of the model in the neighborhood of the origin.

Assumption 4.2 *The shock satisfies $\mathbb{E}|\ln \varepsilon| = \int |\ln z| \boldsymbol{\varphi}(dz) < \infty$.*

Proposition 4.3 *Let $E := (u, f, \varphi, \varrho)$ be given, and let π be an optimal policy. Suppose that Assumptions 2.1–4.2 hold. Define*

$$p := \limsup_{y \to 0} \frac{f(\pi(y))}{y}, \quad q := \liminf_{y \to 0} \frac{f(\pi(y))}{y}.$$

1. *If $p < \exp(-\mathbb{E} \ln \varepsilon)$, then E is globally collapsing to the origin.*
2. *If $q > \exp(-\mathbb{E} \ln \varepsilon)$, then E is globally stable.*

Also, in the light of Lemma 3.4, one might suspect that even in the situation where an economy is globally stable for every ϱ, the stationary distribution will become more and more concentrated around the origin when $\varrho \downarrow 0$. In this connection,

Proposition 4.4 *Let u, f and φ be given. Suppose that (u, f, φ, ϱ) is globally stable for all $\varrho \in (0, 1)$. If $\varrho_n \to 0$, then $\boldsymbol{\psi}_n^* \to \delta_0$ in the weak topology, where $\boldsymbol{\psi}_n^*$ is the stationary distribution corresponding to ϱ_n, and δ_0 is the probability measure concentrated at zero.*

Remark. As δ_0 and $\boldsymbol{\psi}_n^*$ are mutually singular, norm convergence is impossible.

5 Proofs

In the proofs, $L_1(X)$ refers as usual to all integrable Borel functions on given space X, and $C^n(X)$ is the n times continuously differentiable functions.

5.1 Monotonicity

The proof of monotonicity of the optimal policy is as follows.

PROOF. [Proof of Lemma 3.2] Let π be optimal, and take any nonnegative $y \leq y'$. If $y = y'$ then monotonicity is trivial. Suppose the inequality is strict. By way of contradiction, suppose that $\pi(y) > \pi(y')$. Define $c := y - \pi(y)$, $c' := y' - \pi(y')$, and $\hat{c} := \pi(y) - \pi(y') > 0$. Note first that

$$c' - \hat{c} = y' - \pi(y) > y - \pi(y) = c \geq 0. \tag{6}$$

Also, since $c + \hat{c} + \pi(y') = y$, we have

$$u(c) + \varrho \int V[f(\pi(y))z]\boldsymbol{\varphi}(dz) \geq u(c + \hat{c}) + \varrho \int V[f(\pi(y'))z]\boldsymbol{\varphi}(dz),$$

and since $c' - \hat{c} + \pi(y) = y'$,

$$u(c') + \varrho \int V[f(\pi(y'))z]\boldsymbol{\varphi}(dz) \geq u(c' - \hat{c}) + \varrho \int V[f(\pi(y))z]\boldsymbol{\varphi}(dz).$$

$$\therefore \quad u(c') - u(c' - \hat{c}) \geq u(c + \hat{c}) - u(c).$$

As $c' - \hat{c} > c$ by (6), this contradicts the strict concavity of u.

PROOF. [Proof of Lemma 3.3] Pick any $y \geq 0$. Let $k_0 := \pi_0(y)$ and $k_1 := \pi_1(y)$. By definition,

$$u(y - k_0) + \varrho_0 \int V(f(k_0)z)\boldsymbol{\varphi}(dz) \geq u(y - k_1) + \varrho_0 \int V(f(k_1)z)\boldsymbol{\varphi}(dz)$$

and

$$u(y - k_1) + \varrho_1 \int V(f(k_1)z)\boldsymbol{\varphi}(dz) \geq u(y - k_0) + \varrho_1 \int V(f(k_0)z)\boldsymbol{\varphi}(dz).$$

Multiplying the first inequality by ϱ_1 and the second by ϱ_0 and adding gives

$$\varrho_1 u(y - k_0) + \varrho_0 u(y - k_1) \geq \varrho_1 u(y - k_1) + \varrho_0 u(y - k_0).$$

$$\therefore \quad (\varrho_1 - \varrho_0)(u(y - k_0) - u(y - k_1)) \geq 0.$$

$$\therefore \quad \varrho_1 \geq \varrho_2 \implies u(y - k_0) - u(y - k_1) \geq 0 \implies k_1 \geq k_0.$$

PROOF. [Proof of Lemma 3.4] Since u is concave, for any $y > 0$ and any $k \leq y$,

$$u(y - k) \leq u(y) - u'(y)k. \tag{7}$$

Also, since $u(y) \leq M < \infty$ for all y,

$$V(y) := \sup_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(y_t - \pi(y_t))\right] \leq \frac{1}{1 - \varrho} M. \tag{8}$$

Since $\pi(y) = 0$ is feasible,

$$u(y - \pi(y)) + \varrho \int V(f(\pi(y))z)\boldsymbol{\varphi}(dz) \geq u(y) + \varrho \int V(f(0)z)\boldsymbol{\varphi}(dz) = u(y).$$

$$\therefore \quad u(y) - u(y - \pi(y)) \leq \varrho \int V(f(\pi(y))z)\boldsymbol{\varphi}(dz) \leq \frac{\varrho}{1-\varrho} M.$$

Using the bound (7) gives us

$$u'(y)\pi(y) \leq \frac{\varrho}{1-\varrho} M, \quad \forall y > 0.$$

$$\therefore \quad \pi(y) \leq \frac{\varrho}{1-\varrho} \frac{M}{u'(y)} := b(y; \varrho).$$

The function $y \to b(y, \varrho)$ is continuous and converges pointwise to zero as $\varrho \to 0$. The statement follows (uniform convergence on compact sets is by Dini's Theorem).

5.2 The Ramsey–Euler equation

Next Propositions 3.1 and 3.2 are established. We use the following lemma, which can be thought of as a kind of convolution argument designed to verify precisely the conditions necessary for the Ramsey–Euler equation to hold. The proof is rather long, and is relegated to the appendix.

Lemma 5.1 *Let g and h be nonnegative real functions on $\mathbb{R}$. Define*

$$\mu(r) := \int_{-\infty}^{\infty} h(x + r)g(x)\, dx. \tag{9}$$

Consider the following conditions:

(i) $g \in L_1(\mathbb{R}) \cap C^1(\mathbb{R})$, $g' \in L_1(\mathbb{R})$

(ii) *h is bounded*

(iii) *h is nondecreasing*

(iv) *h is absolutely continuous on compact intervals*

(v) *h' is bounded on compact subsets of $\mathbb{R}$,*

where h' is defined as the derivative of h when it exists and zero elsewhere. If (i) *and* (ii) *hold, then $\mu \in C^1(\mathbb{R})$, and*

$$\mu'(r) = -\int_{-\infty}^{\infty} h(x + r)g'(x)\, dx. \tag{10}$$

If, in addition, (iii)–(v) *hold, then μ' also has the representation*

$$\mu'(r) = \int_{-\infty}^{\infty} h'(x + r)g(x)\, dx. \tag{11}$$

Remark. Note that higher order derivatives are immediate if g has high order derivatives that are all integrable. In the first part of the proof, where differentiability and the representation $\mu'(r) = -\int h(x+r)g'(x)dx$ are established we do not use nonnegativity of g—it may be any real function. So now suppose that g is twice differentiable, and that $g'' \in L_1(\mathbb{R})$. Then by applying the same result, this time using g' for g, differentiability of μ' is verified.

To prove Proposition 3.1, the following preliminary observation is important.

Lemma 5.2 *Assume the hypotheses of Proposition 3.1, and let V be the value function. The map $k \mapsto \int V[f(k)z]\boldsymbol{\varphi}(dz)$ is continuously differentiable on the interior of $\mathbb{R}_+$.*

PROOF. By a simple change of variable,

$$\int_0^\infty V[f(k)z]\varphi(z)dz = \int_{-\infty}^\infty V[\exp(\ln f(k) + x)]\varphi(e^x)e^x\, dx.$$

Let $h(x) := V[\exp(x)]$, $g(x) := \varphi(e^x)e^x$, and let μ be defined as in (9). Then $\int V[f(k)z]\varphi(z)dz = \mu[\ln f(k)]$. Regarding μ, conditions (i) and (ii) of Lemma 5.1 are satisfied by (U2), (S1) and (S2). Hence $\int V[f(k)z]\varphi(z)dz$ is continuously differentiable as claimed.

Now let us consider the interiority result.

PROOF. [Proof of Proposition 3.1, Part 1.] Pick any $y > 0$. Consider first the claim that $\pi(y) \neq 0$. Suppose instead that $0 \in \Sigma(y)$, so that

$$V(y) = u(y) - \varrho \int V[f(0)z]\boldsymbol{\varphi}(dz) = u(y), \tag{12}$$

where we have used $u(0) = 0$ in (U2). Define also

$$V_\xi := u(y - \xi) + \varrho \int V[f(\xi)z]\boldsymbol{\varphi}(dz), \tag{13}$$

where ξ is a positive number less than y. By (F3), there exists a $\delta > 0$ such that $f(\xi) > \xi$ whenever $\xi < \delta$. Therefore,

$$V_\xi \geq u(y - \xi) + \varrho \int V(\xi z)\boldsymbol{\varphi}(dz), \quad \forall \xi < \delta. \tag{14}$$

In addition, $V \geq u$ everywhere on $\mathbb{R}_+$. Using this bound along with (12) and (14) gives

$$0 \leq \frac{V(y) - V_\xi}{\xi} \leq \frac{u(y) - u(y - \xi)}{\xi} - \varrho \int \frac{u(\xi z)}{\xi}\boldsymbol{\varphi}(dz), \quad \forall \xi < \delta. \tag{15}$$

Take a sequence $\xi_n \downarrow 0$. If $H_n(z) = u(\xi_n z)/\xi_n$, then $H_n \geq 0$ on $\mathbb{R}_+$ and $H_{n+1}(z) \geq H_n(z)$ for all z and all n. Moreover $\lim_{n\to\infty} H_n = \infty$ almost everywhere. By the Monotone Convergence Theorem, then,

$$\lim_{n\to\infty} \int \frac{u(\xi_n z)}{\xi_n}\boldsymbol{\varphi}(dz) = \int \infty\, \boldsymbol{\varphi}(dz) = \infty,$$

which induces a contradiction in (15).

Now consider the claim that $\pi(y) \neq y$. Let

$$v(k) := u(y-k) + w(k), \quad w(k) := \varrho \int V[f(k)z]\boldsymbol{\varphi}(dz), \quad k \in [0, y].$$

If $y \in \Sigma(y)$, then for all positive ε,

$$0 \leq \frac{v(y) - v(y-\varepsilon)}{\varepsilon} = -\frac{u(\varepsilon)}{\varepsilon} + \frac{w(y) - w(y-\varepsilon)}{\varepsilon}. \tag{16}$$

Since $w(k)$ is differentiable at y (Lemma 5.2), the second term on the right-hand side converges to a finite number as $\varepsilon \downarrow 0$. In this case clearly there will be a contradiction of inequality (16). This completes the proof that $y \notin \Sigma(y)$.

PROOF. [Proof of Proposition 3.1, Part 2.] Regarding the existence of left and right derivatives, pick any $y > 0$, any $\xi_n \downarrow 0$, $\xi_n > 0$, and any optimal policy π. By monotonicity, $\pi(y+\xi_n)$ converges to some limit k_+, and the value k_+ is independent of the choice of sequence (ξ_n). Moreover, upper hemi-continuity of π implies that k_+ is maximal at y. It follows from this and interiority of optimal policies that $0 < k_+ < y$ and

$$V(y) = u(y - k_+) + \varrho \int V[f(k_+)z]\boldsymbol{\varphi}(dz).$$

Also, for all $n \in \mathbb{N}$,

$$\begin{aligned} V(y+\xi_n) &= u(y + \xi_n - \pi(y+\xi_n)) + \varrho \int V[f(\pi(y+\xi_n))z]\boldsymbol{\varphi}(dz) \\ &\geq u(y - k_+ + \xi_n) + \varrho \int V[f(k_+)z]\boldsymbol{\varphi}(dz). \end{aligned}$$

$$\therefore \quad u(y - k_+ + \xi_n) - u(y - k_+) \leq V(y+\xi_n) - V(y), \quad \forall n \in \mathbb{N}.$$

On the other hand, since $\pi(y+\xi_n) \downarrow k_+ < y$, there exists an $N \in \mathbb{N}$ such that

$$V(y) \geq u(y - \pi(y+\xi_n)) + \varrho \int V[f(\pi(y+\xi_n))z]\boldsymbol{\varphi}(dz), \quad \forall n \geq N.$$

$$\therefore \quad V(y+\xi_n) - V(y) \leq u(y + \xi_n - \pi(y+\xi_n)) - u(y - \pi(y+\xi_n)), \quad \forall n \geq N.$$

$$\therefore \quad V(y+\xi_n) - V(y) \leq u'(y - \pi(y+\xi_n))\xi_n, \quad \forall n \geq N,$$

where the last inequality is by concavity of u. In summary, then,

$$u(y - k_+ + \xi_n) - u(y - k_+) \leq V(y+\xi_n) - V(y) \leq u'(y - \pi(y+\xi_n))\xi_n$$

for all n sufficiently large. Dividing through by $\xi_n > 0$ and taking limits gives $V'_+(y) = u'(y - k_+)$, which is of course finite by $k_+ < y$.[11]

Now consider the analogous argument for V'_-. Let y, (ξ_n) and π be as above. Again, as π is monotone, $\pi(y - \xi_n) \uparrow k_-$, where k_- is independent of the precise sequence

[11] We are using continuity of u', which is guaranteed by Assumption 2.1.

(ξ_n), maximal at y and satisfies $0 < k_- < y$. Since $k_- > 0$, the sequence $\pi(y-\xi_n)$ will be positive for large enough n and we can assume this is so for all n. By maximality,

$$V(y) = u(y - k_-) + \varrho \int V[f(k_-)z]\boldsymbol{\varphi}(dz).$$

Also, since $k_- < y$, there exists an $N \in \mathbb{N}$ with $k_- \leq y - \xi_n$ for all $n \geq N$. Hence, $\forall n \geq N$,

$$\begin{aligned} V(y - \xi_n) &= u(y - \xi_n - \pi(y - \xi_n)) + \varrho \int V[f(\pi(y - \xi_n))z]\boldsymbol{\varphi}(dz) \\ &\geq u(y - k_- - \xi_n) + \varrho \int V[f(k_-)z]\boldsymbol{\varphi}(dz). \end{aligned}$$

$$\therefore \quad u(y - k_- - \xi_n) - u(y - k_-) \leq V(y - \xi_n) - V(y), \quad \forall n \geq N.$$

One the other hand, since $0 < \pi(y - \xi_n) \uparrow k_- < y$,

$$V(y) \geq u(y - \pi(y - \xi_n)) + \varrho \int V[f(\pi(y - \xi_n))z]\boldsymbol{\varphi}(dz), \quad \forall n \in \mathbb{N}.$$

$$\therefore \quad V(y - \xi_n) - V(y) \leq u(y - \xi_n - \pi(y - \xi_n)) - u(y - \pi(y - \xi_n)), \quad \forall n \in \mathbb{N}.$$

$$\therefore \quad V(y - \xi_n) - V(y) \leq -u'(y - \pi(y - \xi_n))\xi_n, \quad \forall n \in \mathbb{N},$$

where again the last inequality is by concavity of u. Putting the inequalities together gives

$$u(y - k_- - \xi_n) - u(y - k_-) \leq V(y - \xi_n) - V(y) \leq u'(y - \pi(y - \xi_n))(-\xi_n)$$

for all n sufficiently large. Dividing through by $-\xi_n$ and taking limits gives $V'_-(y) = u'(y - k_-)$.

PROOF. [Proof of Proposition 3.1, Part 3.] The proof is identical to that given in Mirman and Zilcha (1975, Lemma 1).

PROOF. [Proof of Proposition 3.1, Part 4.] The proof is essentially the same as that of Majumdar, Mitra and Nyarko (1989, Lemma 4). Briefly, it is clear from the proof of Part 2 of Proposition 3.1 that $V'_-(y)$ and $V'_+(y)$ will agree whenever $\Sigma(y)$ is a singleton. If y_1 and y_2 are any two distinct points where Σ is multi-valued, then $\Sigma(y_1)$ and $\Sigma(y_2)$ can intersect at at most one point, otherwise we can construct a non-monotone optimal policy, contradicting Lemma 3.2. It follows that for each y where $\Sigma(y)$ is multi-valued, $\Sigma(y)$ can be allocated a unique rational number.

Next we come to the proof of the Ramsey–Euler equation. We need the following lemma, which was first proved (under different assumptions) by Majumdar, Mitra and Nyarko (1989, Lemma 2A).

Lemma 5.3 *For every compact $K \subset (0, \infty)$, $\inf\{y - \pi(y) : y \in K\}$ is strictly positive.*

PROOF. Suppose to the contrary that on some compact set $K \subset (0, \infty)$, there exists for each n a y_n with $\pi(y_n) > y_n - 1/n$. By compactness (y_n) has a convergent subsequence, and without loss of generality we assume that the whole sequence converges to $y^* \in K$. The bounded sequence $\pi(y_n)$ itself has a convergent subsequence $\pi(y_{n(i)}) \to k^*$ as $i \to \infty$. Since the subsequence $(y_{n(i)})$ converges to y^* too, k^* is optimal at y^* by upper hemicontinuity. But then $y^* - \frac{1}{n(i)} \leq k^* \leq y^*$ for all $i \in \mathbb{N}$. This contradicts the interiority of the optimal policy, which has already been established.

The next lemma is fundamental to our results.

Lemma 5.4 *Define V' to be the derivative of V when it exists and zero elsewhere. For all $k > 0$,*

$$\frac{d}{dk} \int V[f(k)z]\varphi(z)dz = \int V'(f(k)z)f'(k)z\varphi(z)dz.$$

PROOF. We change variables to shift the problem to the real line. Our objective is to apply Lemma 5.1. Let $w(k) := \int V[f(k)z]\varphi(z)dz$. As before, we can use a change of variable to obtain

$$w(k) = \int_{-\infty}^{\infty} V(f(k)e^x)\varphi(e^x)e^x dx = \int_{-\infty}^{\infty} h(x + \ln f(k))g(x)dx,$$

where $g(x) := \varphi(e^x)e^x$ and $h(x) := V(e^x)$. All of the hypotheses of Lemma 5.1 are satisfied.[12] Therefore, using the representation (11),

$$\begin{aligned} w'(k) &= \frac{f'(k)}{f(k)} \int_{-\infty}^{\infty} h'(x + \ln f(k))g(x)dx \\ &= f'(k) \int_{-\infty}^{\infty} V'(e^x f(k))e^x g(x)dx. \end{aligned}$$

Changing variables again gives the desired result:

$$w'(k) = \int_0^{\infty} V'(f(k)z)f'(k)g(\ln z)dz = \int_0^{\infty} V'(f(k)z)f'(k)z\varphi(z)dz.$$

Now the proof of the Ramsey–Euler equation can be completed.

PROOF. [Proof of Proposition 3.2] Evidently $\pi(y)$ solves

$$u'(y - k) - \varrho \frac{d}{dk} \int V[f(k)z]\varphi(z)dz = 0.$$

The result now follows from Lemma 5.4, given that $V'(y) = u'(y - \pi(y))$ Lebesgue almost everywhere.

[12] In particular, h' is bounded on compact sets, because $h'(x) = V'(e^x)e^x$, and $V'(y) = u'(y - \pi(y))$ when it exists (i.e., when the function V' is not set to zero). The latter is bounded on compact sets by Lemma 5.3. Also, V is absolutely continuous because countinuous functions of bounded variation (provided by monotonicity here) fail to be absolutely continuous only if they have infinite derivative on an uncountable set (Saks, 1937, p. 128). This is impossible by Proposition 3.1, Part 4.

In the following discussion let an optimal policy π be given. We simplify notation by defining the map S by $S(y) := f(\pi(y))$. The most important properties of S are that S is nondecreasing and $S(y) = 0$ implies $y = 0$ (see Lemma 3.2 and Proposition 3.1, Part 1). Also, let $\mathscr{D}$ be all $\boldsymbol{\psi} \in \mathscr{P}$ that are absolutely continuous with respect to Lebesgue measure.

Define the Markov operator $\mathbf{M}\colon \mathscr{P} \ni \boldsymbol{\psi} \to \mathbf{M}\boldsymbol{\psi} \in \mathscr{P}$ corresponding to π by

$$(\mathbf{M}\boldsymbol{\psi})(B) = \int \int \mathbb{1}_B[S(y)z]\boldsymbol{\varphi}(dz)\boldsymbol{\psi}(dy). \tag{17}$$

It is immediate from (4) that the sequence of marginal distributions $(\boldsymbol{\psi}_t)$ for income satisfies $\boldsymbol{\psi}_{t+1} = \mathbf{M}\boldsymbol{\psi}_t$ for all $t \geq 0$.

We note the following facts, which are easy to verify. First, if $\boldsymbol{\psi}(\{0\}) = 0$, then $\mathbf{M}\boldsymbol{\psi} \in \mathscr{D}$. It follows immediately that $\mathbf{M}(\mathscr{D}) \subset \mathscr{D}$, and that $(\boldsymbol{\psi}_t)$, the sequence of marginal distributions for income, satisfies $\boldsymbol{\psi}_t \in \mathscr{D}$ for all $t \geq 1$. Also, if $\boldsymbol{\psi} \in \mathscr{D}$, then the simple change of variable $y' = S(y)z$ gives

$$(\mathbf{M}\boldsymbol{\psi})(B) = \int \int_B k(y, y')dy'\boldsymbol{\psi}(dy), \tag{18}$$

where dy' is of course integration with respect to Lebesgue measure, and

$$k(y, y') := \varphi\left(\frac{y'}{S(y)}\right)\frac{1}{S(y)}. \tag{19}$$

It is immediate from Definition 4.1 that a steady state is a fixed point of the Markov operator in $\mathscr{P}$ which puts zero mass on $\{0\}$. Since such distributions are mapped into $\mathscr{D}$ by $\mathbf{M}$, when a steady state $\boldsymbol{\psi}^*$ exists it must be in $\mathscr{D}$.

The next lemma is just elementary manipulation of the definitions.

Lemma 5.5 *Let π be a fixed optimal policy, and let $\mathbf{M}$ be the corresponding Markov operator.*

1. *The economy is globally stable in the sense of Definition 4.1 if and only if there is a unique $\boldsymbol{\psi}^* \in \mathscr{D}$ with $\mathbf{M}\boldsymbol{\psi}^* = \boldsymbol{\psi}^*$ and $\mathbf{M}^t\boldsymbol{\psi} \to \boldsymbol{\psi}^*$ in norm as $t \to \infty$ for every $\boldsymbol{\psi} \in \mathscr{D}$.*
2. *The economy is globally collapsing to the origin in the sense of Definition 4.2 if and only if $\mathbf{M}^t\boldsymbol{\psi}([a, \infty)) \to 0$ for every $\boldsymbol{\psi} \in \mathscr{D}$ and every $a > 0$.*

PROOF. [Proof of Lemma 4.1] By Corollary 3.1, any pair of optimal policies is equal almost everywhere. Inspection of (18) and (19) indicates that they will have identical Markov operators on $\mathscr{D}$, in the sense that if $\mathbf{M}$ corresponds to one optimal policy and $\mathbf{M}'$ to another, then $\mathbf{M}\boldsymbol{\psi} = \mathbf{M}'\boldsymbol{\psi}$ for all $\boldsymbol{\psi} \in \mathscr{D}$. The rest of the proof of Part 1 follows immediately from Lemma 5.5. The proof of Part 2 is similar.

PROOF. [Proof of Proposition 4.1] Let $\mathbf{M}$ be the Markov operator corresponding to π, and let k be as in (19). Consider the following two conditions:

(i) $\mathbf{M}\boldsymbol{\psi}$ dominates the Lebesgue measure ($\mathbf{M}\boldsymbol{\psi}$-null sets are Lebesgue null) for all $\boldsymbol{\psi} \in D$.

(ii) $\forall \hat{y} > 0$, $\exists \varepsilon > 0$ and $\eta \geq 0$ with $\int \eta(x)dx > 0$ and

$$k(y, y') \geq \eta(y')\mathbf{1}_{(\hat{y}-\varepsilon, \hat{y}+\varepsilon)}(y), \quad \forall\, y, y'.$$

Here by Rudnicki (1995, Theorem 2 and Corollary 3), (i) and (ii) imply the Foguel Alternative; in particular that either $\mathbf{M}$ has a unique fixed point $\boldsymbol{\psi}^* \in \mathscr{D}$ and $\mathbf{M}^t\boldsymbol{\psi} \to \boldsymbol{\psi}^*$ in norm for all $\boldsymbol{\psi} \in \mathscr{D}$, or alternatively $\mathbf{M}$ is sweeping with respect to the compact sets, so that $\lim_{t\to\infty} \mathbf{M}^t\boldsymbol{\psi}([a, b]) = 0$ for any $\boldsymbol{\psi} \in \mathscr{D}$ and any $0 < a < b < \infty$. In the light of Lemma 5.5, then, to prove Proposition 4.1 it is sufficient to check (i), (ii) and, in addition,

$$\lim_{b\to\infty} \limsup_{t\to\infty} \int \mathbf{M}^t\boldsymbol{\psi}([\,b, \infty)) = 0, \quad \forall \boldsymbol{\psi} \in \mathscr{D}, \tag{20}$$

where (20) demonstrates that sweeping occurs not just with respect to any interval $[a, b]$, $a > 0$, but in fact to any interval $[a, \infty)$.

Condition (i) is immediate from the assumption that φ is everywhere positive, in light of (18) and (19). Regarding condition (ii), pick any $\hat{y} > 0$ and any ε such that $\hat{y} - \varepsilon > 0$. Also let $0 < \gamma_0 < \gamma_1 < \infty$. Define

$$\delta_0 := \frac{\gamma_0}{S(\hat{y} + \varepsilon)}, \quad \delta_1 := \frac{\gamma_1}{S(\hat{y} - \varepsilon)}.$$

Note that $\inf_{z\in[\delta_0,\delta_1]} \varphi(z) > 0$ by (S1) and strict positivity. Set

$$r := \frac{\inf_{z\in[\delta_0,\delta_1]} \varphi(z)}{S(\hat{y} + \varepsilon)}, \quad \eta := r\mathbb{1}_{[\gamma_0,\gamma_1]}.$$

Then η has the required properties.

Regarding (20), from (F2) there exists a $\alpha \in (0, 1)$ and $m < \infty$ such that $S(y) \leq \alpha y + m$ for all $y \in \mathbb{R}_+$. Then

$$y_{t+1} \leq (\alpha y_t + m)\varepsilon_t. \tag{21}$$

Since y_t and ε_t are independent and $\mathbb{E}\varepsilon = 1$ we have

$$\mathbb{E}y_{t+1} \leq \alpha\mathbb{E}y_t + m. \tag{22}$$

Using an induction argument gives

$$\mathbb{E}y_t \leq \alpha^t\mathbb{E}y_0 + (1 + \alpha + \cdots + \alpha^{t-1})m \leq \alpha^t\mathbb{E}y_0 + \frac{m}{1-\alpha}. \tag{23}$$

Suppose that $\mathbb{E}y_0 < \infty$. Then from (23) it follows that

$$\limsup_{t\to\infty} \mathbb{E}y_t \leq \frac{m}{1-\alpha}. \tag{24}$$

By the Chebychev inequality, $\mathbf{M}^t\boldsymbol{\psi}([\,b, \infty)) \leq \mathbb{E}y_t b^{-1}$. From (24) it then follows that (20) holds for all $\boldsymbol{\psi}$ with $\mathbb{E}y_0 := \int y\boldsymbol{\psi}(dy) < \infty$. This set (all densities with finite first moments) is norm-dense in $\mathscr{D}$, and $\mathbf{M}$ is an L_1 contraction on $\mathscr{D}$. Together, these facts imply that condition (20) in fact holds for every $\boldsymbol{\psi} \in \mathscr{D}$ (Lasota and Mackey 1994, p. 126).

PROOF. [Proof of Lemma 4.2] Regarding Part 1, let π_0 (resp. π_1) be an optimal policy for E_0 (resp. E_1), let $\mathbf{M}_0$ and $\mathbf{M}_1$ be the corresponding Markov operators and let $(y_t^0)_{t\geq 0}$ and $(y_t^1)_{t\geq 0}$ be the respective income processes. By Lemmas 4.1 and 5.5 it is sufficient to show that for any $\boldsymbol{\psi} \in \mathscr{D}$ and any $a > 0$ we have

$$\lim_{t\to\infty} \mathbf{M}_0^t \boldsymbol{\psi}([\,a,\infty)) = 0. \tag{25}$$

From Lemma 3.3 we have $\pi_1 \geq \pi_0$ pointwise on $\mathbb{R}_+$, so it is clear (by induction) that

$$y_t^1 \geq y_t^0 \text{ pointwise on } \Omega \text{ for any } t.$$

$$\therefore \quad \{y_t^0 \geq a\} \subset \{y_t^1 \geq a\}.$$

$$\therefore \quad \mathbf{M}_0^t \boldsymbol{\psi}([\,a,\infty)) = \mathbf{P}\{y_t^0 \geq a\} \leq \mathbf{P}\{y_t^1 \geq a\} = \mathbf{M}_1^t \boldsymbol{\psi}([\,a,\infty)).$$

By Lemma 5.5 and the hypothesis, the right hand side converges to zero as $t \to \infty$, which proves (25).

PROOF. [Proof of Proposition 4.3] For this proof we set $x_t := \ln y_t$, and define $\alpha := \mathbb{E}\ln\varepsilon$, $\eta := \ln\varepsilon - \alpha$ and $T\colon \mathbb{R} \ni x \to \ln f(\pi(e^x)) + \alpha$, so that $x_{t+1} = T(x_t) + \eta_t$, where $\mathbb{E}\eta_t = 0$.

(Part 1) By the condition, $\limsup_{x\to-\infty}(T(x) - x) < 0$, implying the existence of an $m \in \mathbb{R}$ and $a > 0$ such that $T(x) \leq x - 2a$, for all $x \leq m$.

$$\therefore \quad x_{t+1} \leq x_t + \eta_t - 2a, \quad \forall x_t \leq m.$$

Let $\hat{x}_t := x_t - m$ and $\hat{\eta}_t := \eta_t - a$. Then

$$\hat{x}_{t+1} \leq \hat{x}_t + \hat{\eta}_t - a, \quad \forall \hat{x}_t \leq 0. \tag{26}$$

Define $\Omega_0 := \{\omega \in \Omega \,:\, \sup_{T\geq 0} \sum_{t=0}^T \hat{\eta}_t(\omega) \leq 0\}$. Since $\mathbb{E}\hat{\eta}_t = -a < 0$, it follows that $\mathbf{P}(\Omega_0) > 0$ (Borovkov, 1998, Chapter 11). From (26) we have

$$\hat{x}_t \leq \hat{x}_0 + \hat{\eta}_0 + \cdots + \hat{\eta}_{t-1} - ta \;\text{ for } \omega \in \Omega_0,$$

so if $\mathbf{P}\{\hat{x}_0 \leq 0\} = 1$, then $\mathbf{P}\{x_t \leq -at\} \geq \mathbf{P}(\Omega_0) > 0$ for all t. Since $\{\hat{x}_t \leq -at\} = \{y_t \leq e^{m-at}\}$, we have shown the existence of an initial condition y_0 ($\mathbf{P}\{\hat{x}_0 \leq 0\} = 1$ if y_0 is chosen s.t. $\mathbf{P}\{y_0 \leq e^m\} = 1$) with the property

$$\liminf_{t\to\infty} \mathbf{P}\{y_t \leq c\} = \liminf_{t\to\infty} \boldsymbol{\psi}_t([0,c]) \geq \mathbf{P}(\Omega_0) > 0.$$

But then $\boldsymbol{\psi}_t$ cannot converge in norm to any $\boldsymbol{\psi}^* \in \mathscr{D}$. (If $\boldsymbol{\psi}_t \to \boldsymbol{\psi}^* \in \mathscr{D}$ then $\boldsymbol{\psi}_t([0,c]) \to \boldsymbol{\psi}^*([0,c])$, so choosing $c > 0$ such that $\boldsymbol{\psi}^*([0,c]) < \mathbf{P}(\Omega_0)$ leads to a contradiction.) Therefore the economy is not globally stable, and it follows from Proposition 4.1 that it must be collapsing to the origin.

(Part 2) By the condition, $\liminf_{x\to-\infty}(T(x)-x)>0$, there is an $m\in\mathbb{R}$ and $a>0$ such that $T(x)\geq x+a$ whenever $x\leq m$. Let $\hat{x}:=x-m$ and $\hat{\eta}:=\eta+a$. Then $\hat{x}_{t+1}\geq\hat{x}_t+\hat{\eta}_t$ whenever $\hat{x}_t\leq 0$. Also, since T is nondecreasing, $\hat{x}\geq 0$ implies $T(x)\geq m+a$. Therefore $\hat{x}_t\geq 0 \implies \hat{x}_{t+1}\geq\hat{\eta}_t$.

$$\therefore\quad \hat{x}_{t+1}\geq -\hat{x}_t^- + \hat{\eta}_t \geq -(-\hat{x}_t^- + \hat{\eta}_t)^-, \tag{27}$$

where we are using the standard notation $x^- := -\min(0,x)$ and $x^+ := \max(0,x)$.

Assume to the contrary that the economy is not globally stable, in which case it must be sweeping from the sets $[a,\infty)$, all $a>0$, so that for each $c\in\mathbb{R}$ we have

$$\lim_{t\to\infty}\mathbf{P}\{\hat{x}_t\leq c\}=1. \tag{28}$$

Let us introduce now the process (z_t) defined by $z_0 := -\hat{x}_0^-$, $z_{t+1} := -(z_t+\hat{\eta}_t)^-$. By (27) we have $z_t\leq\hat{x}_t$ for all t. Since $\hat{\eta}_0$ is $\mathbf{P}$-integrable, there is an $L>0$ such that $\mathbb{E}(\hat{\eta}_0-L)^+<a/3$. Let y_0 be chosen so that $\hat{x}_0$ is also integrable. Then $\mathbb{E}|z_0|<\infty$, and in fact $\mathbb{E}|z_t|<\infty$ for all t. From (28) and $z_t\leq\hat{x}_t$ we have

$$\lim_{t\to\infty}\mathbf{P}\{z_t\leq -L\}=1.$$

Choose t_0 so that $\mathbf{P}\{z_t>-L\}<a/(3L)$ when $t\geq t_0$. Since $z_t\leq 0$, then, $t\geq t_0$ implies $\mathbb{E}(z_t+L)^+<a/3$. Therefore,

$$\begin{aligned}\mathbb{E}z_{t+1} = -\mathbb{E}(z_t+\hat{\eta}_t)^- &= \mathbb{E}(z_t+\hat{\eta}_t)-\mathbb{E}(z_t+\hat{\eta}_t)^+\\ &\geq \mathbb{E}z_t+\mathbb{E}\hat{\eta}_t-\mathbb{E}(z_t+L)^+-\mathbb{E}(\hat{\eta}_t-L)^+\\ &> \mathbb{E}z_t+\frac{a}{3},\end{aligned}$$

which contradicts $z_t\leq 0$ for all t.

PROOF. [Proof of Proposition 4.4] By the Portmanteau Theorem (Shiryaev, 1996, Theorem III.1.1), $\boldsymbol{\psi}_n^*\to\delta_0$ weakly if and only if

$$\liminf_{n\to\infty}\boldsymbol{\psi}_n^*(G)\geq\delta_0(G)\quad\text{for every open set } G\subset\mathbb{R}_+.$$

Here by "open" we refer to the relative topology on $\mathbb{R}_+$. Evidently the above condition is equivalent to $\lim_n\boldsymbol{\psi}_n^*(G)=1$ for all open G containing 0, which in turn is equivalent to

$$\lim_{n\to\infty}\boldsymbol{\psi}_n^*([a,\infty))=0,\quad\forall a>0.$$

Take (π_n) to be any sequence of optimal policies corresponding to $\varrho_n\to 0$. Let (y_t^n) be the Markov chain generated by π_n and fixed initial distribution $y_0\sim\boldsymbol{\psi}_0$ (i.e., $y_{t+1}^n=f(\pi_n(y_t^n))\varepsilon_t$). Here $y_0=y_0^n$ is chosen so that $\mathbb{E}y_0<\infty$.

Consider the probability that y_t^n exceeds a. For each real R we have

$$\begin{aligned}\mathbf{P}\{y_t^n\geq a\} = \mathbf{P}(\{y_t^n\geq a\}&\cap\{y_{t-1}^n\leq R\})\\ &+\mathbf{P}(\{y_t^n\geq a\}\cap\{y_{t-1}^n>R\}).\end{aligned} \tag{29}$$

Consider the second term. We claim that

$$\forall r > 0,\ \exists R \in \mathbb{R} \ \text{ s.t. } \sup_{n \in \mathbb{N}} \sup_{t \geq 0} \mathbf{P}\{y_t^n > R\} < r. \tag{30}$$

To see this, fix $r > 0$, and pick any $n \in \mathbb{N}$. Define a sequence (ξ_t) of random variables on $(\Omega, \mathscr{F}, \mathbf{P})$ by $\xi_0 = y_0$, $\xi_{t+1} = (\alpha\xi_t + \beta)\varepsilon_t$, where $y \mapsto \alpha y + \beta$ is an affine function dominating f on $\mathbb{R}_+$ and satisfying $\alpha < 1$ (see the comment after Assumption 2.2). From the definition of y_t^n, the fact that $\pi_n(y) \leq y$ and $f(y) \leq \alpha y + \beta$, it is clear that $y_t^n \leq \xi_t$ pointwise on Ω for all t, and hence

$$\forall R \in \mathbb{R}, \quad \{y_t^n > R\} \subset \{\xi_t > R\}.$$

$$\therefore \quad \mathbf{P}\{y_t^n > R\} \leq \mathbf{P}\{\xi_t > R\}, \quad \forall t \geq 0. \tag{31}$$

Since ξ_t and ε_t are independent, $\mathbb{E}\xi_{t+1} = \alpha\mathbb{E}\xi_t + \beta$. It follows that

$$\mathbb{E}\xi_t \leq \alpha^t \mathbb{E}\xi_0 + \frac{\beta}{1-\alpha} \leq \mathbb{E}\xi_0 + \frac{\beta}{1-\alpha}$$

for all t. Since $\mathbb{E}\xi_0 = \mathbb{E}y_0 < \infty$ we see that $\mathbb{E}\xi_t \leq C$ for all t, where C is a finite constant. By the Chebychev inequality, then,

$$\mathbf{P}\{\xi_t > R\} \leq \frac{\mathbb{E}\xi_t}{R} \leq \frac{C}{R}, \quad \forall t \geq 0. \tag{32}$$

Combining (31) and (32) gives $\mathbf{P}\{y_t^n > R\} < C/R$ for all t and n. Since R is arbitrary the claim (30) is established.

Our objective was to bound the second term in (29). So fix $r > 0$. By (30) we can choose R so large that

$$\mathbf{P}\{y_t^n \geq a\} = \mathbf{P}(\{y_t^n \geq a\} \cap \{y_{t-1}^n \leq R\}) + \frac{r}{2} \tag{33}$$

for all t and all n. It remains to bound the first term. Let $(\boldsymbol{\psi}_t^n) \subset \mathscr{P}$ be the sequence of marginal distributions associated with (y_t^n). From the well-known expression for the finite dimensional distribution of Markov chains on measurable rectangles (e.g., Shiryaev, 1996, Theorem II.9.2) we have

$$\mathbf{P}(\{y_t^n \geq a\} \cap \{y_{t-1}^n \leq R\}) \\ = \int_0^R \int_a^\infty \varphi\left(\frac{y'}{f(\pi_n(y))}\right) \frac{1}{f(\pi_n(y))} dy' \boldsymbol{\psi}_{t-1}(dy).$$

A change of variable gives

$$\int_a^\infty \varphi\left(\frac{y'}{f(\pi_n(y))}\right) \frac{1}{f(\pi_n(y))} dy' = \boldsymbol{\varphi}([a/f(\pi_n(y)), \infty)).$$

From the proof of Lemma 3.4, we know that π_n is dominated by an increasing function b_n which converges pointwise to zero. Therefore $f \circ \pi_n$ is dominated by $f \circ b_n$, again an increasing function, which must by continuity of f converge pointwise and hence uniformly to zero on $[0, R]$. Combining this with the fact that $a > 0$ and $\boldsymbol{\varphi}$ is a finite measure, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\boldsymbol{\varphi}([a/f(\pi_n(y)), \infty)) < \frac{r}{2}, \quad \forall y \in [0, R].$$

But then

$$\mathbf{P}(\{y_t^n \geq a\} \cap \{y_{t-1}^n \leq R\}) \leq \int_0^R \frac{r}{2} \boldsymbol{\psi}_{t-1}(dy) \leq \frac{r}{2}.$$

Using this inequality together with (29) and (33), we conclude that for all $r > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ and $t \geq 0$ implies $\mathbf{P}\{y_t^n \geq a\} = \boldsymbol{\psi}_t^n([a, \infty)) < r$. Since $\boldsymbol{\psi}_t^n \to \boldsymbol{\psi}_n^*$ in norm it follows that $\boldsymbol{\psi}_t^n([a, \infty)) \to \boldsymbol{\psi}_n^*([a, \infty))$ in $\mathbb{R}$ as $t \to \infty$, so that $\boldsymbol{\psi}_n^*([a, \infty)) \leq r$ is also true. That is, $\lim_{n \to \infty} \boldsymbol{\psi}_n^*([a, \infty)) = 0$, as was to be proved.

A Appendix

First we need the following lemma regarding continuity of translations in L_1, which is well-known.

Lemma A.1 *Let g be in $L_1(\mathbb{R})$. If $\tau(t) := \|g(x-t) - g(x)\|$, then τ is bounded on $\mathbb{R}$, and $\tau(t) \to 0$ as $t \to 0$.*

Now define the real number $\mu'(r)$ to be $-\int h(x+r) g'(x)\, dx$, which is clearly finite. By the Fundamental Theorem of Calculus,

$$\begin{aligned} \mu(r+t) - \mu(r) - \mu'(r)t &= \int h(x+r)(g(x-t) - g(x) + g'(x)t)\, dx \\ &= -t \int h(x+r) \int_0^1 (g'(x-ut) - g'(x))\, du\, dx. \end{aligned}$$

Taking absolute values, using (ii) and Fubini's theorem,

$$\left| \frac{\mu(r+t) - \mu(r)}{t} - \mu'(r) \right| \leq M \int_0^1 \int |g'(x-ut) - g'(x)|\, dx\, du \tag{A.1}$$

for some M. By Lemma A.1, $\int |g'(x-ut) - g'(x)|\, dx$ is uniformly bounded in u and converges to zero as $t \to 0$ for each $u \in [0, 1]$. By Lebesgue's Dominated Convergence Theorem the term on the right hand side of (A.1) then goes to zero and

$$\mu'(r) = -\int h(x+r) g'(x)\, dx.$$

Regarding continuity of the derivative, we have

$$\begin{aligned} |\mu'(r+t) - \mu'(r)| &\leq \int h(x) |g'(x-r-t) - g'(x-r)|\, dx \\ &\leq M \int |g'(x-t) - g'(x)|\, dx. \end{aligned}$$

Continuity now follows from Lemma A.1.

Next we argue that under (iii)–(v),

$$\mu'(r) = \int h'(x+r)g(x)\,dx \tag{A.2}$$

is also valid. To begin, define $\mu_h'(r)$ to be the right hand side of (A.2). This number exists in $\mathbb{R}$, because

$$h'(x+r) = \liminf_{t\downarrow 0} \frac{h(x+r+t) - h(x+r)}{t}$$

almost everywhere by either (iii) or (iv), and hence

$$\begin{aligned}\mu_h'(r) = \int \liminf_{t\downarrow 0} \frac{h(x+r+t) - h(x+r)}{t} g(x)dx \\ \leq \liminf_{t\downarrow 0} \int \frac{h(x+r+t) - h(x+r)}{t} g(x)dx = \mu'(r).\end{aligned}$$

Here the inequality follows from the assumption that h is increasing, which gives nonnegativity of the difference quotient, and Fatou's Lemma.

By (iv) the Fundamental Theorem of Calculus applies to h, and

$$\begin{aligned}\mu(r+t) - \mu(r) - \mu_h'(r)t &= \int (h(x+t) - h(x) - h'(x)t)g(x-r)\,dx \\ &= t\int\int_0^1 (h'(x+ut) - h'(x))g(x-r)\,dx\,du.\end{aligned}$$

Some simple manipulation gives

$$\mu_h'(r) = \mu'(r) - \lim_{t\to 0}\int\int_0^1 (h'(x+ut) - h'(x))g(x-r)\,dx\,du.$$

Thus it is sufficient to now show that

$$\lim_{t\to 0}\int_0^1\int |h'(x+ut) - h'(x)|g(x-r)\,dx\,du = 0.$$

The inner integral is bounded independent of u, because it is less than

$$\int h'(x+ut)g(x-r)\,dx + \int h'(x)g(x-r)\,dx \leq \mu'(r+ut) + \mu'(r),$$

which is bounded for $u \in [0,1]$ by continuity of μ'. Thus by Lebesgue's Dominated Convergence Theorem we need only prove that

$$\lim_{t\to 0}\int |h'(x+ut) - h'(x)|g(x-r)\,dx = 0.$$

Adding and subtracting appropriately, this integral is seen to be less than

$$\begin{aligned}\int |h'(x+ut)g(x-r+ut) - h'(x)g(x-r)|dx \\ + \int |h'(x+ut)g(x-r) - h'(x+ut)g(x-r+ut)|\,dx.\end{aligned} \tag{A.3}$$

Consider the first integral in the sum. By Lemma A.1, we can choose a $\delta_0 > 0$ such that $|t| \leq \delta_0$ implies

$$\int |h'(x+ut)g(x-r+ut) - h'(x)g(x-r)|dx < \frac{\varepsilon}{3}.$$

The second integral in the sum can be written as

$$\int_{|x|\leq R} |h'(x+ut)g(x-r) - h'(x+ut)g(x-r+ut)|\, dx \\ + \int_{|x|\geq R} |h'(x+ut)g(x-r) - h'(x+ut)g(x-r+ut)|\, dx.$$

By the usual property of L_1 functions, we can choose R such that the integral over $|x| \geq R$ is less than $\varepsilon/3$ for all t with $|t| \leq \delta_0$.

To summarize the results so far, we have $|t| \leq \delta_0$ implies

$$\int |h'(x+ut) - h'(x)|g(x-r)\, dx \\ < \frac{2\varepsilon}{3} + \int_{|x|\leq R} |h'(x+ut)g(x-r) - h'(x+ut)g(x-r+ut)|\, dx.$$

Finally, since h' is bounded on compact sets,

$$h'(x+ut) \leq M, \quad \forall x, t \text{ with } |x| \leq R,\ |t| \leq \delta_0.$$

Therefore $|t| \leq \delta_0$ implies

$$\int |h'(x+ut) - h'(x)|g(x-r)\, dx \\ < \frac{2\varepsilon}{3} + M \int |g(x-r) - g(x-r+ut)|\, dx.$$

By Lemma A.1 there is a $\delta_1 > 0$ such that

$$M \int |g(x-r) - g(x-r+ut)|\, dx < \frac{\varepsilon}{3}$$

whenever $|t| < \delta_1$. Now setting $\delta := \delta_0 \wedge \delta_1$ gives

$$|t| \leq \delta \implies \int |h'(x+ut) - h'(x)|g(x-r)\, dx < \varepsilon$$

as required.

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