Log-Linearization of Stochastic Economic Models ⋆

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Abstract

This paper studies formally the common practice of log-linearizing stochastic economic models, making precise the conditions under which stability of the original model can be inferred from that of the linearized model. A transformation to recover the stochastic equilibrium of the former from that of the latter is provided.

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1 Introduction

Consider a stochastic economic system with the stylized representation

$$x_{t+1} = f(x_t, \varepsilon_t), \quad t = 0, 1, \ldots, \tag{OR}$$

where x is a vector of state variables taking values in space X, f is an arbitrary function with $\operatorname{rng} f \subset X$, and (ε_t) is a sequence of shocks. Given (OR), the

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researcher seeks to characterize dynamics of the sequence (x_t). Ideally, the distribution of x_t will converge to a unique limiting distribution independent of x_0 as $t \to \infty$. This distribution is called the dynamic equilibrium (in what follows, equilibrium), or stochastic steady state. [1]

In the economic literature, a common approach to the analysis of stochastic dynamics is via linearization. However, it must be remembered that the linearized system is auxiliary to the analysis: *it is valuable only to the extent that it provides insight into the dynamic properties of the true model* (OR).

In this connection, we emphasize that for stochastic systems such as (OR), it is not in general legitimate to infer such properties as existence, uniqueness and stability of equilibrium from similar properties as they may or may not occur in the linear version. For example, the presence of noise makes local analysis used to justify linearization problematic.

In the log-linear case it seems likely that the stochastic dynamics will be similar, given that the log transformation is bijective and continuous. Here that intuition is verified, with some caveats regarding assumptions on the distribution of the shock. A formula for translating the equilibrium of the linearized system back into the equilibrium of the original system is provided.

A caveat regarding our results is that many linearization techniques differ from log-linearization in that they are not in general representable as bijective transformations from one state space to another. A typical example is linearization by Taylor expansion. Such linearizations do not meet the conditions of our theorem.

Section 2 states the problem. Section 3 states results. Section 4 gives an application to the multisector macroeconomic model of Long and Plosser (1983). Section 5 gives proofs.

[1] The problem of existence and stability for such equilibria has been studied by many authors, including Brock and Mirman (1972), Futia (1982), Hopenhayn and Prescott (1992), Bhattacharya and Majumdar (2001) and Stachurski (2002).

2 Outline of the Problem

If $(U, \mathscr{A})$ is some measurable space, then $\mathscr{P}(U) = \mathscr{P}(U, \mathscr{A})$ denotes the probabilistic measures on $(U, \mathscr{A})$. If U has a topology, then $bC(U)$ is the continuous bounded real functions on U, and $\mathscr{A}$ is always identified with $\mathscr{B}_U$, the Borel sets on U. In this case, $\mathscr{P}(U)$ is endowed with one of two topologies. The first is the probabilists topology of weak convergence, which is the smallest topology on $\mathscr{P}(U)$ making the set of real functions $\{\mu \mapsto \int g d\mu : g \in bC(U)\}$ continuous. The other is the norm topology induced by the total variation norm $\| \cdot \|_{TV}$.[2]

The model (OR) obeys the following assumptions. Time is discrete. The state space X is any topological space. The shocks ε_t take values in measurable space $(W, \mathscr{W})$. The map $f \colon X \times W \to X$ is $(\mathscr{B}_X \otimes \mathscr{W}, \mathscr{B}_X)$-measurable. The sequence $(\varepsilon_t)_{t \geq 0}$ is independent and identically distributed on probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with common distribution $\psi \in \mathscr{P}(W)$.

The model generates a Markov process $(x_t)_{t \geq 0}$ on $(\Omega, \mathscr{F}, \mathbb{P})$. Let the distribution of x_t be denoted $\mu_t \in \mathscr{P}(X)$. Following Brock and Mirman (1972), Futia (1982, p. 377), Stokey et al. (1989) and others, a (dynamic) equilibrium for the economy (OR) is a $\mu_* \in \mathscr{P}(X)$ satisfying

$$\mu_*(B) = \int_X \left[\int_W \mathbb{1}_B[f(x, z)] \psi(dz) \right] \mu_*(dx), \quad \forall B \in \mathscr{B}_X. \tag{1}$$

The equality (1) implies that if x_t has distribution μ_*, then so does x_{t+1}, as indeed do all subsequent x_{t+j}. The equilibrium is unique if there exists no other element of $\mathscr{P}(X)$ satisfying (1). The equilibrium is called globally stable in the weak (respectively, norm) topology if μ_t converges to μ_* weakly (respectively, in norm) as $t \to \infty$ for every initial condition x_0.

3 Results

Log-linearization involves a bijective transformation applied to both sides of (OR). One hopes that stability of the transformed system will be easier to

[2] See, for example, Stokey et al. (1989), Aliprantis and Border (1999, Ch. 14) or any other standard text for definitions.

establish. However it still remains to show that the transformed system and the original system have the "same" dynamics.

We prove a version of this result using a general bijective transformation, of which the log function is a special case. Precisely, let L be a $(\mathscr{B}_X, \mathscr{B}_{\hat{X}})$-measurable one-to-one map from X onto topological space $\hat{X}$. Applying L to both sides of (OR) gives the transformed system

$$\hat{x}_{t+1} = L[f(L^{-1}(\hat{x}_t), \varepsilon_t)], \quad \hat{x}_s := L(x_s). \tag{TR}$$

Equilibria (elements of $\mathscr{P}(\hat{X})$) and stability for (TR) are defined analogously to the case of (OR).

Corresponding to L there is a map $\mathbf{L}$ from $\mathscr{P}(X)$ to $\mathscr{P}(\hat{X})$ defined by

$$\mathbf{L}\mu(B) = \mu \circ L^{-1}(B) = \mu(L^{-1}(B)), \quad B \in \mathscr{B}_X. \tag{2}$$

In other words, $\mathbf{L}$ maps $\mu \in \mathscr{P}_X$ into its image measure under L. Clearly $\mathbf{L}$ is itself a bijection.

Our main result follows. The proof is in Section 5.

Proposition 3.1 *If $\mu_* \in \mathscr{P}(X)$ is an equilibrium for* (OR)*, then $\mathbf{L}\mu_* \in \mathscr{P}(\hat{X})$ is an equilibrium for the transformed economy* (TR)*. Conversely, if $\hat{\mu}_* \in \mathscr{P}(\hat{X})$ is an equilibrium for* (TR)*, then $\mathbf{L}^{-1}\hat{\mu}_* \in \mathscr{P}(X)$ is an equilibrium for* (OR)*. Furthermore, if $\hat{\mu}_* \in \mathscr{P}(\hat{X})$ is globally stable for* (TR) *in the norm topology, then $\mathbf{L}^{-1}\hat{\mu}_*$ is globally stable for* (OR) *in the norm topology. If, in addition to the above hypotheses, L is continuous with continuous inverse, then $\mathbf{L}^{-1}\hat{\mu}_*$ is globally stable for* (OR) *in the weak topology whenever $\hat{\mu}_*$ is globally stable for* (TR) *in the weak topology.*

4 Application

A well-known study using log-linearization is the Real Business Cycle model of Long and Plosser (1983). They study an infinite horizon, representative agent economy with n sectors. Production is according to the Cobb-Douglas technology

$$y_{i,t+1} = \lambda_{it} \ell_{it}^{b_i} x_{i1t}^{a_{i1}} \times \cdots \times x_{int}^{a_{in}}, \quad i = 1, \ldots n, \tag{3}$$

where y_i is output of commodity i, ℓ_i is labor allocated to sector i, x_{ij} is the amount of commodity j used in the production of good i, a_{ij} is the input-output elasticity, and λ_{it} is the time t value of the i-th sector-specific shock. The vector of shocks λ_t is uncorrelated and identically distributed over time.[3]

Production is assumed to be constant returns to scale. In particular,

$$b_i, a_{ij} > 0; \quad b_i + \sum_{j=1}^{n} a_{ij} = 1, \quad i = 1, \ldots, n. \tag{4}$$

After specifying preferences and constraints, the model is solved by dynamic programming to yield the system

$$\begin{aligned} y_{1,t+1} &= \gamma_{1t} y_{1t}^{a_{11}} \times \cdots \times y_{nt}^{a_{1n}} \\ &\vdots \qquad\qquad \vdots \\ y_{n,t+1} &= \gamma_{nt} y_{1t}^{a_{n1}} \times \cdots \times y_{nt}^{a_{nn}}, \end{aligned} \tag{LP}$$

where $\gamma_t = (\gamma_{it})_{i=1}^n$ is a vector of strictly positive shocks depending on λ_t and the parameters.

Following Long and Plosser, we can convert (LP) into a linear form and study stability of the latter. By virtue of Proposition 3.1, this will imply conditions under which (LP) is itself stable. Specifically,

Proposition 4.1 *Let γ_t be the vector of sectoral shocks in* (LP). *If the expectation $\mathbb{E}\,\|(\ln \gamma_{1t}, \ldots, \ln \gamma_{nt})\|$ is finite, then the economy* (LP) *has a unique equilibrium in $\mathscr{P}(\mathbb{R}^n_{++})$ which is globally stable in the norm topology.*

Remark The restriction $\mathbb{E}\,\|(\ln \gamma_{1t}, \ldots, \ln \gamma_{nt})\| < \infty$ in Proposition 4.1 enforces small left- and right-hand tails on the distributions of the sectoral shocks. Without a small left-hand tail assumption the economy may collapse to zero output. For this reason, a simple finite mean assumption such as $\mathbb{E}\,\|\gamma_t\|$ is *not* in general sufficient.

[3] The objective of Long and Plosser was to generate fluctuations in time series consistent with the business cycle from a general equilibrium framework and without assuming correlated noise.

PROOF. The linearized version of (LP) is

$$\hat{y}_{t+1} = A\hat{y}_t + L(\gamma_t), \quad \hat{y} \in \mathbb{R}^n, \tag{5}$$

where $\hat{y} \in \mathbb{R}^n$ is log income, $A := (a_{ij})$ is the matrix of input-output elasticities, and $L\colon \mathbb{R}^n_{++} \to \mathbb{R}^n$ is defined by $L(x) := (\ln x_i)_{i=1}^n$. Linear stochastic difference equations such as (5) are well understood. It is known that if $\mathbb{E}\,\|L(\gamma_t)\| < \infty$ and $\sup_{\|x\|=1} \|Ax\|/\|x\| =: \varrho < 1$, then (5) has a unique and (norm) globally stable equilibrium distribution $\hat{\mu}_*$ in $\mathscr{P}(\mathbb{R}^n)$.[4]

That $\varrho < 1$ follows from (4). The finiteness of $\mathbb{E}\,\|L(\gamma_t)\|$ is given in the statement of the proposition. It now follows from Proposition 3.1 that the original model (LP) has a unique, globally stable equilibrium $\mathbf{L}^{-1}\hat{\mu}_* \in \mathscr{P}(\mathbb{R}^n_{++})$.

5 Proofs

It remains to prove Proposition 3.1. The method is as follows. First, we recall the notion of topological conjugacy between dynamical systems. Conjugate systems have identical dynamics. Since the notion of conjugacy is defined for deterministic rather than stochastic systems, our next step is to convert (OR) and (TR) into deterministic self-mappings on spaces of probability measures. We then show that these deterministic versions are topologically conjugate.

Let U and $\hat{U}$ be topological spaces. Consider the two dynamical systems

$$x_{t+1} = g(x_t), \quad g\colon U \to U, \tag{6}$$

$$\hat{x}_{t+1} = \hat{g}(\hat{x}_t), \quad \hat{g}\colon \hat{U} \to \hat{U}. \tag{7}$$

Suppose that, corresponding to g and $\hat{g}$, there exists a homeomorphism (continuous bijection with continuous inverse) $\mathbf{L}$ from U into $\hat{U}$ such that g and $\hat{g}$ commute with $\mathbf{L}$ in the sense that $\hat{g} = \mathbf{L} \circ g \circ \mathbf{L}^{-1}$ on U. Then (6) and (7) are said to be *topologically conjugate*.

In this case, $x_* \in U$ is a fixed point of g on U if and only if $\mathbf{L}(x_*) \in \hat{U}$ is a fixed point of $\hat{g}$ on $\hat{U}$, and $\lim_{t\to\infty} g^t(x) = x_*$ for all $x \in U$ if and only if

[4] See, for example, Meyn and Tweedie (1993).

$\lim_{t\to\infty} \hat{g}^t(\hat{x}) = \mathbf{L}(x_*)$ for all $\hat{x} \in \hat{U}$.[5] These results are well-known and easy to verify.

Thus to prove Proposition 3.1, it is necessary to rewrite (OR) and (TR) in the form of (6) and (7), whereby equilibria will be fixed points of some suitable mapping, and show that they commute with a homeomorphism.

Regarding the first task, consider again the system (OR). As before, let $\mu_t \in \mathscr{P}(X)$ be the marginal distribution of the random variable x_t. Define an operator $\mathbf{Q}$ from $\mathscr{P}(X)$ into itself by

$$(\mathbf{Q}\mu)(B) = \int_X \left[\int_W \mathbb{1}_B[f(x,z)]\psi(dz) \right] \mu(dx), \quad B \in \mathscr{B}_X. \tag{8}$$

Given $\mathbf{Q}$, it is well-known (c.f., e.g., Stokey et al., 1989, p. 219) that μ_t and μ_{t+1} are connected by $\mu_{t+1} = \mathbf{Q}\mu_t$. In the present context $\mathbf{Q}$ is called the *Foias operator* corresponding to (OR).[6]

From $\mu_{t+1} = \mathbf{Q}\mu_t$ it must be that $\mu_t = \mathbf{Q}^t \mu_0$, where μ_0 is the distribution of x_0, and $\mathbf{Q}^t$ is defined by $\mathbf{Q}^t := \mathbf{Q} \circ \mathbf{Q}^{t-1}$ and $\mathbf{Q}^1 := \mathbf{Q}$. In light of (1), probability $\mu_* \in \mathscr{P}$ is an equilibrium for the system (OR) if and only if $\mathbf{Q}\mu_* = \mu_*$. The equilibrium is globally stable if and only if $\mathbf{Q}^t\mu_0 \to \mu_*$ as $t \to \infty$ for all $\mu_0 \in \mathscr{P}(X)$.

All of the above extends analogously to (TR). We let $\hat{\mathbf{Q}}$ be the Foias operator corresponding to this law of motion. That is, $\hat{\mathbf{Q}} \colon \mathscr{P}(\hat{X}) \to \mathscr{P}(\hat{X})$, where

$$(\hat{\mathbf{Q}}\mu)(B) = \int_{\hat{X}} \left[\int_W \mathbb{1}_B\{L[f(L^{-1}(\hat{x}), z)]\}\psi(dz) \right] \mu(d\hat{x}), \quad B \in \mathscr{B}_{\hat{X}}.$$

The two systems (OR) and (TR) can now be represented as

$$\mu_{t+1} = \mathbf{Q}\mu_t, \quad \mathbf{Q} \colon \mathscr{P}(X) \to \mathscr{P}(X), \tag{9}$$

$$\hat{\mu}_{t+1} = \hat{\mathbf{Q}}\hat{\mu}_t, \quad \hat{\mathbf{Q}} \colon \mathscr{P}(\hat{X}) \to \mathscr{P}(\hat{X}). \tag{10}$$

Note that (9) and (10) are in the same form as (6) and (7). Thus to complete the proof of Proposition 3.1 we need to exhibit a homeomorphism $\mathbf{L} \colon \mathscr{P}(X) \to \mathscr{P}(\hat{X})$ such that $\mathbf{Q} = \mathbf{L}^{-1} \circ \hat{\mathbf{Q}} \circ \mathbf{L}$ on $\mathscr{P}(X)$.

[5] For map h, h^t denotes the t-th composition of h with itself.

[6] See, for example, Lasota and Mackey (1994, Chapter 12).

A suitable candidate for $\mathbf{L}$ is the map defined in (2). Before continuing, we recall that (see, e.g., Aliprantis and Border, 1999, Theorem 12.46) if $\mu \in \mathscr{P}(X)$, and h is any bounded $\mathscr{B}_{\hat{X}}$-measurable real function on $\hat{X}$, then $h \circ L \colon X \to \mathbb{R}$ is $\mathscr{B}_X$-measurable, and

$$\int_{\hat{X}} h \, d\mathbf{L}\mu = \int_X h \circ L \, d\mu. \tag{11}$$

Lemma 5.1 *The map* $\mathbf{L}$ *is a bijection and a homeomorphism in the norm topology. If* L *is a homeomorphism, then* $\mathbf{L}$ *is also a homeomorphism in the weak topology.*

PROOF. Norm continuity of $\mathbf{L}$ is immediate from the bound $\|\mathbf{L}\mu - \mathbf{L}\nu\|_{TV} \leq \|\mu - \nu\|_{TV}$, $\forall \mu, \nu \in \mathscr{P}(X)$, which is easy to establish. Regarding weak continuity of $\mathbf{L}$, let (μ_α) be a net in $\mathscr{P}(X)$ converging to $\mu \in \mathscr{P}(X)$. Pick any h in $bC(\hat{X})$. If L is continuous, then

$$\int_{\hat{X}} h \, d\mathbf{L}\mu_\alpha = \int_X h \circ L \, d\mu_\alpha \to \int_X h \circ L \, d\mu = \int_{\hat{X}} h \, d\mathbf{L}\mu,$$

since $h \circ L \in bC(X)$. The proof of continuity of $\mathbf{L}^{-1}$ is similar.

To complete the proof of Proposition 3.1 we need only show that

Lemma 5.2 *The relation* $\mathbf{Q} = \mathbf{L}^{-1} \circ \hat{\mathbf{Q}} \circ \mathbf{L}$ *holds on* $\mathscr{P}(X)$.

PROOF. We prove the equivalent statement $(\hat{\mathbf{Q}} \circ \mathbf{L}\mu)(B) = (\mathbf{L} \circ \mathbf{Q}\mu)(B)$ for all $\mu \in \mathscr{P}(X)$ and all $B \in \mathscr{B}_{\hat{X}}$. From the definitions of the Foias operators,

$$\begin{aligned}
(\hat{\mathbf{Q}}\mathbf{L}\mu)(B) &= \int_{\hat{X}} \left[\int_W \mathbb{1}_B\{L[f(L^{-1}(\hat{x}), z)]\}\psi(dz) \right] \mathbf{L}\mu(d\hat{x}) \\
&= \int_X \left[\int_W \mathbb{1}_B\{L[f(x, z)]\}\psi(dz) \right] \mu(dx) \quad (\because \text{ by } (11)) \\
&= \int_X \left[\int_W \mathbb{1}_{L^{-1}(B)}[f(x, z)]\psi(dz) \right] \mu(dx) = (\mathbf{L}\mathbf{Q}\mu)(B).
\end{aligned}$$

This establishes the lemma and hence of Proposition 3.1.

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