Discrete Time Models in Economic Theory

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Abstract

The paper exposits a number of key dynamic models from the field of economics. The models share the feature that evolution of the system is consistent with the optimizing behavior of rational economic agents. Specifically, the laws of motion that we investigate are solutions to discrete time dynamic decision processes. Even with rational agents and simple economic environments, a variety of complex behaviors are shown to obtain.

1 Introduction

The primary concern of economics as a science is allocation of scarce resources among several alternative and competing uses. In this connection, it is important to bear in mind that resources must be allocated not only contemporaneously, but also across time. The planning involved occurs in each period after observation of the current state. Therefore decisions of intertemporal allocation are naturally described by discrete-time dynamical functions.

These dynamical processes are not defined by arbitrarily chosen laws of motion. Agents' decisions are made according to such concerns as profit maximization by firms, utility maximization by households and social welfare maximization by the policy maker. In formulating these plans, forward looking agents consider the affect of their actions on the time path of state variables. The dynamical systems used in economic theory are thus obtained as solutions to such intertemporal optimization problems.

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The foundational models in this area are those that replicate growth through the accumulation of productive capital. In economics, interest in the theory of growth has been revived during the last fifteen years. The reason for the resurrection of growth research is that the framework of intertemporal optimization has been found to explain a much wider range of phenomena than was previously believed. Using results from the field of non-linear dynamical systems, it has been shown that intertemporal optimization theory can provide new explanations for business cycles and for international differences in growth and development.

In the economic literature, historically there have been two types of explanations for observed fluctuations in the level of economic activity. One type of explanation has been built on the view that fluctuations are caused by factors that are exogenous to economic systems. For example, agricultural production can be affected by weather, consumers tastes may be influenced by intangible fads, or government policies may change erratically. In such environments the market plays the role of a filter, passing random shocks to economic variables, which constantly deviate from their trends. These deviations are regarded as business cycles; the field of research is referred to as the "real business cycle" literature.

The other type of explanation views fluctuations as phenomena endogenous to economic systems. This view dominated the literature prior to the 1960s.[3] Those studies in turn were based on the Keynesian premise that certain regularities exist in the relationships between major macroeconomic variables. It has become clear, however, that such a premise is inconsistent with the rational and well-informed behavior of economic agents, and this inconsistency has contributed to a shift of research focus towards what is now known as neoclassical economics. The latter is built on the foundations of rational, optimizing behavior.

The purpose of the neoclassical methodology is not of course to deny the occurrence of irrational behavior by actors in the real economy. The force of its argument comes from its logical consistency and the potential for prediction. Without *a priori* restrictions on human market and strategic behavior, models consistent with any outcome can be constructed. Theories that cannot preclude any possibility are not useful for structuring academic debate.

If the neoclassical paradigm is accepted, however, it remains unclear whether or not the fundamental structure of an economy itself—without the influence of external noise—may explain business fluctuations in the neoclassical framework. Work on non-linear dynamics from the late 1970s and early 1980s appears to have answered this question in the affirmative, renewing widespread

[3] See, for example, Harrod [15], Samuelson [37], or Kaldor [17].

interest in the endogenous explanation of economic fluctuations. [4]

In the first part of this paper, a number of deterministic discrete time models are presented. All treat by intertemporal optimization the fundamental problem of savings and consumption. Some models are associated with asymptotically stable dynamics, while others exhibit cycles and other complex behavior.

In addition, some basic results for stochastic dynamic models are also discussed. Stochastic models are important because of their obvious connections with the empirical literature. In addition, they form the foundations of the so-called real business cycles mentioned above, which remain an active research area. Among the results listed here, particular emphasis is placed on the implications of monotone decision rules for asymptotic stochastic behavior.

2 Aggregative models of economic growth

2.1 The basic framework

The basic premise of the aggregative model can be described as follows: in each period t, a single homogeneous output, Y_t, is produced from the two homogeneous input factors labor, L_t, and capital, K_t. The technically efficient possibilities for production are summarized by an aggregate production function $F(K_t, L_t)$ which exhibits constant returns to scale, positive marginal productivity, and decreasing marginal rate of substitution.

Constant returns—scalar multiplying the vector of inputs multiplies output by the same amount—is motivated by a replication argument. Diminishing returns to individual inputs when others are held fixed seems a natural assumption, and generates convexities that are central to both optimization and dynamic properties.

Because of constant returns to scale, the output-labor ratio $y_t = Y_t/L_t$ is given by

$$y_t = f(k_t), \tag{1}$$

[4] Benhabib and Nishimura [3] demonstrate the appearance of periodic cycles in along optimal paths of capital accumulation in a model where agents are fully rational and perfectly informed. Benhabib and Day [1], and Grandmont [13] observe the possibilities of chaotic dynamics in models in which agents are rational and informed in certain limited manners. Boldrin and Montrucchio [5], Deneckere and Pelikan [9], and Nishimura and Yano [31] demonstrate the existence of a chaotic optimal path of capital accumulation in a model with fully rational and perfectly informed agents.

where $k_t = K_t/L_t$ is the capital-labor ratio and $f(k) = F(k, 1)$. Regarding f, it is assumed that

Assumption 1 *$f : [0, \infty) \to \mathbb{R}$ is continuous everywhere, and twice continuously differentiable on $(0, \infty)$.*

Further,

Assumption 2 *$f(0) = 0$, $f' > 0$, $f'' < 0$, $\lim_{k \downarrow 0} f'(k) = \infty$, $\lim_{k \uparrow \infty} f'(k) = 0$.*

The labor force is assumed to be constant, and capital stock depreciates at positive rate δ. Per capita output may be allocated between consumption and gross investment. Denoting per capita consumption by c_t this implies

$$y_t = c_t + k_{t+1} - (1 - \delta)k_t. \tag{2}$$

The initial per capita capital stock k_0 is historically given. Social welfare over the infinite planning period is presumed to be represented by the functional

$$\sum_{t=0}^{\infty} \varrho^t u(c_t), \tag{3}$$

where $\varrho \in (0, 1)$ is the discount factor. Thus, social welfare is the discounted sum of period-wise utility of per capita consumption.

Assumption 3 *$u : [0, \infty) \to \mathbb{R}$ is continuous, increasing, and twice continuously differentiable on $(0, \infty)$.*

Assumption 4 *On $(0, \infty)$, $u' > 0$ and $u'' < 0$. Also, $\lim_{c \downarrow 0} u'(c) = \infty$ and $\lim_{c \uparrow \infty} u'(c) = 0$.*

A sequence of stocks $(k_t)_{t=0}^{\infty}$ is called a feasible path from k_0 if it satisfies the condition $0 \le k_{t+1} \le f(k_t) + (1 - \delta)k_t$ for all $t \ge 0$. For each feasible path there is a corresponding sequence of consumption rates $(c_t)_{t=0}^{\infty}$ determined by (1) and (2). A feasible path is called an interior path if $c_t > 0$ holds for all $t \ge 0$, and it is called stationary if $k_t = k$ for all $t \ge 0$ and some constant $k \ge 0$. An optimal path from k_0 is a feasible path from k_0 that maximizes the objective function (3). By the strict concavity of the utility and production functions we can show that the optimal solution from a given k_0 is unique.

An interior path will be called an Euler path if it satisfies the discrete-time Euler equation

$$u'(c_{t-1}) - \varrho u'(c_t)[f'(k_t) + 1 - \delta] = 0. \tag{4}$$

If an interior path is optimal then it must be an Euler path. By substituting (1) and (2) into (4), the Euler equation becomes a second order difference equation; given k_0 there are infinitely many paths that satisfy it. To distinguish the unique optimum path from the other Euler paths we need an additional optimality condition. In fact it is known that under the assumptions stated above, any Euler path which satisfies the transversality condition

$$\lim_{t\to\infty} \varrho^t u'(c_t)[f'(k_t) + 1 - \delta]k_t = 0$$

is an optimal path (See McKenzie [23]).

2.2 Main results

Because of Assumption 4, every optimal path from a positive initial stock $k_0 > 0$ is an interior path and, consequently, it must be an Euler path. A stationary optimal path $(k, k, k, \ldots)$ with $k > 0$ must satisfy the Euler equation (4) and, hence,

$$\varrho^{-1} = f'(k) + 1 - \delta. \tag{5}$$

A solution to (5) is called a steady state.

Local behavior of the solutions around the steady state may be determined by the characteristic equation

$$\lambda^2 \varrho u''(c^*) - \lambda \left[(1+\varrho)u''(c^*) + \varrho u'(c^*) f''(k^*) \right] + u''(c^*) = 0$$

evaluated at that point. This polynomial equation has two roots, the product of which is $\varrho^{-1} > 1$. Evidently it can never have two roots inside the unit circle. Also, the left-hand side is equal to $u''(c^*) < 0$ when $\lambda = 0$ and equal to $-\varrho u'(c^*) f''(k^*) > 0$ when $\lambda = 1$. Evidently there is always one positive root inside the unit circle. This implies that the steady state is locally a saddle-point. The root inside the unit circle corresponds to the optimal solution, because the path converging to the steady state satisfies the transversality condition. Since that root is positive convergence must be monotone.

The above argument is limited to local dynamics of solutions of the Euler equation (4). However the following global result from the maximizing problem (1)–(3) can be proved. It is the discrete-time version of a result originally due to Cass [7] and Koopmans [18].

Theorem 1 *Consider the model defined by* (1)–(3)*. Under Assumptions 1–4 there exists a unique steady state k^*. Moreover, any optimal path from $k_0 > 0$ is monotone and converges to k^*.*

3 Two-sector models

3.1 The basic framework

In this section the aggregative model of Section 2 is extended to the two-sector model. There are two goods: a pure consumption good, C, and a pure capital good, K. Each sector uses both capital and labor as inputs. Capital input must be made one period prior to the production of output. Labor input is made in the same period as output is produced. Denote by $F_C(K_C, L_C)$ and $F_K(K_K, L_K)$ the production functions of sectors C and K, respectively, where K_i and L_i denote the factor inputs in sector $i \in \{C, K\}$. The production functions are assumed to be increasing in each argument, concave and homogeneous of degree one.

The labor endowment of the economy is constant. Without loss of generality we normalize it to 1. Denote by c_t and y_t the time t per capita outputs of sectors C and K, respectively. Thus we have

$$c_t = F_C(K_{C,t-1}, L_{C,t}), \tag{6}$$
$$y_t = F_K(K_{K,t-1}, L_{K,t}). \tag{7}$$

Moreover, denote by k_{t-1} the aggregate capital input:

$$K_{C,t} + K_{K,t} = k_t. \tag{8}$$

The output of the capital good sector, y_t, represents the gross accumulation of capital;

$$y_t = k_t - (1-\delta)k_{t-1}, \tag{9}$$

where $\delta \in (0,1)$ is the rate of depreciation. Since the total labor force in the economy has been normalized to 1,

$$L_{C,t} + L_{K,t} = 1. \tag{10}$$

As before, $u(c)$ is the representative consumer's contemporaneous utility when he consumes c units of the consumption good. With these notations, the two-sector optimal growth model is described by the maximization problem

$$\begin{aligned} &\text{maximize} \quad \sum_{t=1}^{\infty} \varrho^t u(c_t) \\ &\text{subject to} \quad k_0 = \bar{k}_0 \text{ and constraints (6)–(10),} \end{aligned} \tag{11}$$

where as before $\varrho \in (0,1)$ is the discount factor.

In order to analyze the dynamics of the above model it is convenient to express, for each given amount of capital input k, the trade-off between the two outputs by $c = T(k, y)$. That is,

$$T(k,y) = \max F_C(K_C, L_C) \qquad \text{subject to} \begin{cases} F_K(K_K, L_K) = y, \\ L_C + L_K = 1, \\ K_C + K_K = k. \end{cases} \tag{12}$$

The domain of the function T is $\Omega := \{(k, y) \mid k \geq 0, 0 \leq y \leq F_K(k, 1)\}$. With this definition, the optimal growth model (11) can be transformed as follows:

$$\begin{aligned} &\text{maximize} \ \sum_{t=1}^{\infty} \varrho^t U(k_{t-1}, k_t) \\ &\text{subject to} \ \ k_0 = \bar{k}_0 \text{ and } 0 \leq k_t \leq F_K(k_{t-1}, 1) + (1-\delta)k_{t-1}, \end{aligned}$$

where $U(x, z) := u(T(x, z-(1-\delta)x))$ is called a reduced form utility function.

3.2 Optimal cycles

In this section we assume that the period-wise utility function is linear [i.e., $u(c) = c$] and that capital fully depreciates within one period ($\delta = 1$). The reduced form utility function is then identical to the social production function. That is, $U(k, y) = T(k, y)$. Even if the utility function is linear, it can be shown that the optimal path is still unique under plausible conditions on the production functions.

The Euler equation in the two-sector optimal growth model is

$$U_2(k_{t-1}, k_t) + \varrho U_1(k_t, k_{t+1}) = 0, \tag{13}$$

where $U_1(k, y) = \partial U(k, y)/\partial k$ and $U_2(k, y) = \partial U(k, y)/\partial y$. Any path satisfying the Euler equation and the transversality condition

$$\lim_{t \to \infty} \varrho^t k_t U_1(k_t, k_{+1})$$

is known to be optimal.

The steady state k^* corresponds to a stationary solution $(k^*, k^*, k^*, \ldots)$ of (13). The local behavior around k^* is determined by the roots of the characteristic equation

$$\varrho U_{12}(k^*, k^*)\lambda^2 + [\varrho U_{11}(k^*, k^*) + U_{22}(k^*, k^*)]\lambda + U_{21}(k^*, k^*) = 0. \tag{14}$$

evaluated at the steady state.

As in the aggregated model, the product of the two roots is equal to $\varrho^{-1} > 1$. Therefore at least one root is outside the unit circle. However, unlike the aggregated model, the other root of (14) is not necessarily inside the unit circle. If ϱ is sufficiently small then both roots may be outside the unit circle. On the other hand, since there exists always a unique optimal path starting from k_0, there must still be a path that satisfies the transversality condition. We will characterize its behavior below.

Consider the case in which the production functions in both sectors have the so-called Cobb-Douglas form

$$F_C(K_C, L_C) = K_C^{\alpha} L_C^{1-\alpha}, \quad 0 < \alpha < 1; \tag{15}$$
$$F_K(K_K, L_K) = K_K^{\beta} L_K^{1-\beta}, \quad 0 < \beta < 1. \tag{16}$$

From the first order conditions of (12), we have $(K_K/L_K)/(K_C/L_C) = [\beta/(1-\beta)]/[\alpha/(1-\alpha)]$. Hence

$$(K_K/L_K) - (K_C/L_C) \begin{cases} > 0 & \text{if } \beta > \alpha, \\ < 0 & \text{if } \beta < \alpha. \end{cases} \tag{17}$$

Here K_i/L_i is called the *factor intensity* of sector i, and the left-hand side of (17) is called the factor intensity difference. If $\beta > \alpha$, the production of consumption goods is more labor intensive than the production of capital goods. If $\beta < \alpha$, the converse is true.

In the two sector model with Cobb-Douglas production functions and linear utility, the sign of the cross partial derivative $U_{12}(x, y)$ is determined by the factor intensity difference of the consumption good sector and the capital good sector (Benhabib and Nishimura [3]). This fact, together with the relation (17), implies that

$$U_{12}(x, y) \begin{cases} > 0 & \text{if } \beta > \alpha, \\ < 0 & \text{if } \beta < \alpha. \end{cases} \tag{18}$$

We know that given an initial capital stock, there exists a unique optimal path. Therefore optimal paths in this model must be described by a difference equation of the form $k_{t+1} = h(k_t)$. The function h is called the optimal policy function.

Benhabib and Nishimura [3] have shown that the sign of the cross partials of the reduced form utility function determines whether $h(k_t)$ is increasing or decreasing. This, together with equation (18), implies that, in the case $\beta > \alpha$, the graph of the optimal policy function h is strictly increasing whenever it lies in the interior of Ω. Analogously, if $\alpha > \beta$, then the graph of h is strictly decreasing on the interior of Ω. In the case of $\alpha > \beta$, the optimal policy function h becomes a unimodal map, because the graph of h increases along the boundary of Ω and decreases in the interior of Ω. [5]

The global dynamics of the two sector model with Cobb-Douglas production technology and total capital depreciation ($\delta = 1$) is studied in Nishimura and Yano [33]. For this economy the steady state value is

$$k^* = \frac{\alpha(1-\beta)}{\beta\Big[(1-\alpha) + \varrho\delta(\alpha-\beta)\Big]}(\varrho\beta)^{1/(1-\beta)}, \tag{19}$$

and the roots of (14) are

$$\lambda_1 = \frac{\beta-\alpha}{1-\alpha}, \quad \lambda_2 = \frac{1-\alpha}{\varrho(\beta-\alpha)}. \tag{20}$$

If $\alpha > \beta$, then both roots are negative. The following theorem of Nishimura and Yano [33] gives conditions for the instability of the steady state and thus for the existence of an optimal cycle of period 2:

Theorem 2 *For the economy described by the production functions* (15) *and* (16)*, linear utility function $u(c) = c$, and $\delta = 1$, the following is true. If $\alpha > \beta$ and also*

$$\varrho < \frac{1-\alpha}{\alpha-\beta} < 1,$$

then the steady state k^* given in (19) *is totally unstable, and there exists an optimal path which is periodic of period 2.*

4 Optimal chaos

Boldrin and Montrucchio [5] and Deneckere and Pelikan [9] have provided a constructive method to give examples of two-sector optimal growth models in which the optimal policy function is given by the logistic function $h(x) = 4x(1-x)$. The latter is characterized by chaotic paths. While these results are interesting, however, the examples also show that such optimal chaos requires a sufficiently small discount factor. It is therefore quite natural to ask if there is a general relation between the size of the discount factor and the dynamic complexity of optimal paths. Initial results in this direction were derived by Sorger [40,41]. Subsequently, Sorger [42], Mitra [25], Nishimura and Yano [34],

[5] A function $h\colon [a,b] \to [a,b]$ is called unimodal if there exists $\bar{x} \in [a,b]$ such that h is strictly increasing on $[a,\bar{x}]$ and strictly decreasing on $[\bar{x},b]$.

and Mitra and Sorger [26,27] have shown that there exists a sharp upper bound on the set of discount factors that are compatible with chaotic optimal paths.

4.1 Basic concepts

Three possible definitions of complicated dynamics in systems of the form $x_{t+1} = h(x_t)$ are first discussed. Here, x_t is the state of the economy at time t (for example the capital-labor ratio) and $h : [0,1] \mapsto [0,1]$ is a continuous function which encodes dynamic properties, such as technology and market structure.

We say that the dynamical system $x_{t+1} = h(x_t)$ exhibits ergodic chaos if there exists an absolutely continuous probability measure μ on the interval $[0,1]$ which is invariant and ergodic under h. Here, absolutely continuity means existence of Radon-Nikodym derivative with respect to the Lebesgue measure. Invariance of μ under h means that $\mu\{x \in [0,1] \,|\, h(x) \in B\} = \mu B$ for all measurable $B \subseteq [0,1]$. An invariant measure μ is said to be ergodic if, in addition, for every measurable set $B \subseteq [0,1]$ which satisfies $\{x \in [0,1] \,|\, h(x) \in B\} = B$ we have either $\mu B = 0$ or $\mu B = 1$.

We say that the dynamical system $x_{t+1} = h(x_t)$ exhibits geometric sensitivity if there is a real constant $\gamma > 0$ such that the following is true: for any $\tau = 0, 1, 2, \ldots$ there exists $\varepsilon > 0$ such that for all $x, y \in [0,1]$ with $|x - y| < \varepsilon$ and for all $t \in \{0, 1, \ldots, \tau\}$ it holds that

$$|h^{(t)}(x) - h^{(t)}(y)| \geq (1+\gamma)^t |x - y|.$$

Geometric sensitivity implies that small perturbations of the initial conditions are magnified at a geometric rate over arbitrary but finite time periods. Of course, the geometric magnification cannot last indefinitely because the state space $[0,1]$ of the dynamical system is bounded. Note also that geometric sensitivity implies that there is no stable periodic path of the dynamical system.

Finally, we say that the dynamical system $x_{t+1} = h(x_t)$ exhibits topological chaos if there exists a p-periodic solution for all sufficiently large integers p and there exists an uncountable invariant set $S \subseteq [0,1]$ containing no periodic points such that

$$\liminf_{t \to \infty} |h^{(t)}(x) - h^{(t)}(y)| = 0 < \limsup_{t \to \infty} |h^{(t)}(x) - h^{(t)}(z)|$$

holds, whenever $x \in S$, $y \in S$, and either $x \neq z \in S$ or z is a periodic point. The set S is called a scrambled set. The condition displayed above says that any two trajectories starting in the scrambled set move arbitrarily close to each other but do not converge to each other or to any periodic orbit.[6]

4.2 Dynamic linear programming

In this subsection we introduce the dynamic linear programming problem that is equivalent to an economic growth problem frequently studied by economists. Dynamic linear programming can be treated in the standard LP framework by adding a time structure.[7] In order to demonstrate the existence of chaotic solutions to such a problem, we need to focus on the case in which the solutions to an LP problem can be described by an autonomous system. For this reason, it is necessary to work with an infinite time horizon LP model.

Take the following LP problem of choosing a sequence (x_t) to maximize the functional $\sum_{t=1}^{\infty} \varrho^{t-1} p' x_t$ subject to linear constraint

$$\begin{pmatrix} A & 0 & 0 & \cdots \\ -B & A & 0 & \cdots \\ 0 & -B & A & \cdots \\ 0 & 0 & -B & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} \leq \begin{pmatrix} Bx_0 + d \\ d \\ d \\ d \\ \vdots \end{pmatrix} . \tag{21}$$

Here the discount factor ϱ is a number between 0 and 1, A and B are $m \times n$ matrices of non-negative components, x_t, d and p are $n \times 1$ matrices of non-negative components, and p' is the transpose of p. The intended interpretation of this problem is to maximize the objective function $\sum_{t=1}^{\infty} \varrho^{t-1} p' x_t$, which is the discounted sum of $p' x_t$ over the time periods $t = 1, 2, \ldots$, under the recursive constraints $Ax_t \leq Bx_{t-1} + d$ and with the initial condition $x_0 = x$.

The question arises as to whether or not such an optimal program can lead to a chaotic dynamical system. In order to deal with this issue, the following two questions must be addressed. First, under what conditions is in fact the optimal program (21) a dynamical system in the standard sense, described by a single-valued function? Second, under what condition is the resulting dynamical system chaotic?

As an example of (21), take the following LP problem with parameters a_{11},

[6] See Li and Yorke [21], Sarkovskii [38] and Devaney [8] for additional details of results on chaos.
[7] See Dorfman, Samuelson and Solow [10].

a_{12}, a_{21}, a_{22} and k_0 all positive, and $0 < \varrho < 1$.

$$\max_{(c_1,k_1,c_2,k_2,\ldots)\geq 0} \sum_{t=1}^{\infty} \varrho^{t-1} c_t \quad \text{s.t.} \quad \begin{array}{ll} \text{(i)} & a_{11}c_t + a_{12}k_t \leq 1 \quad\quad t = 1, 2, \ldots; \\ \text{(ii)} & a_{21}c_t + a_{22}k_t \leq k_{t-1} \; t = 1, 2, \ldots; \\ \text{(iii)} & k_0 = k. \end{array} \tag{22}$$

The solutions to (22) can be described by a generalized dynamical system. To this end, for each $(k_{t-1}, k_t) \geq 0$, define $c(k_{t-1}, k_t)$ as the maximum value of $c_t \geq 0$ satisfying conditions (i) and (ii) of (22).

Theorem 3 *For each $k \geq 0$, there is a non-empty subset $H(k)$ of $[0, \infty)$ such that if $(c_1, k_1, c_2, k_2, \ldots)$ is a solution to (22), then*

$$k_t \in H(k_{t-1}), t = 1, 2, \ldots, \tag{23}$$

with $k_0 = k$, and

$$c_t = c(k_{t-1}, k_t). \tag{24}$$

We call the system associated with H a generalized optimal dynamical system. If, in particular, H is a function, we call it an optimal dynamical system. Nishimura and Yano [36] demonstrate that H can in fact be a chaotic optimal dynamical system.

4.3 Two-sector Leontief model

Let us now return to two-sector optimal growth models which are formally equivalent to dynamic linear programming problems as discussed in the previous section. Suppose that the two sectors have the following "Leontief" type technology:

$$F_C(K_C, L_C) = \min\{K_C, L_C\}, \tag{25}$$
$$F_K(K_K, L_K) = \lambda \min\{K_K, L_K/b\}, \tag{26}$$

where $\lambda > \varrho^{-1}$ and $b > 1$. Note that $b > 1$ implies that the capital good sector is more labor intensive than the consumption good sector. We still assume that the utility function is linear and that capital fully depreciates in one period. In this case the maximization problem (11) can have multiple solutions which in general cannot be described by an optimal policy function. Nishimura and Yano [32,35] prove, however, that optimal paths are described by an optimal policy function if the parameter values are suitably chosen. Furthermore, they show that, under certain parameter restrictions, the optimal policy function is expansive and unimodal.[8] Let us now describe this result. Set $\gamma := b - 1$ and

$$h(k) := \begin{cases} \lambda k & \text{if } 0 \le k \le 1/(\gamma+1), \\ -(\lambda/\gamma)(k-1) & \text{if } 1/(\gamma+1) \le k \le 1. \end{cases} \tag{27}$$

Under the assumption $\lambda/(1+\gamma) \le 1$, the function h maps the closed interval $I = [0, \lambda/(1+\gamma)]$ onto itself. For all practical purposes, we may therefore restrict our attention to the interval I and treat h as a function from I onto itself. Nishimura and Yano [35] prove the following.

Theorem 4 *Let h_I be the function defined in* (27) *restricted to the interval $I = [0, \lambda/(1+\gamma)]$. Suppose that the parameters λ, ϱ, and γ satisfy*

$$0 < \varrho < 1 \ , \ \gamma > 0 \ , \ \varrho\lambda > 1 \ \text{ and } \ \gamma + 1 > \lambda. \tag{28}$$

Then optimal paths of the two-sector model with linear utility function and Leontief production functions (25) *and* (26) *satisfy the equation $k_{t+1} = h_I(k_t)$, provided that one of the following two conditions hold:*

(i) $\lambda \le \gamma$, *and*

(ii) $\gamma < \lambda \le \min\{(\gamma + \sqrt{\gamma^2 + 4\gamma})/2, (-1 + \sqrt{1+4\gamma})/(2\varrho)\}$.

Under condition (i) the decreasing portion of h_I has slope larger than or equal to -1. More specifically, if (i) is satisfied with strict inequality, then the positive fixed point of the difference equation $k_{t+1} = h_I(k_t)$ is globally asymptotically stable. If, instead, (i) is satisfied with equality, then every optimal solution from $k > 0$ converges to a period-two cycle, except for the unique path that corresponds to the positive fixed point.

Under condition (ii), the decreasing portion of h_I has slope smaller than -1. In this case, h_I is expansive and unimodal. It has been shown that these two properties imply that the dynamical system $k_{t+1} = h_I(k_t)$ exhibits ergodic chaos and geometric sensitivity. Nishimura and Yano [35] show that the set of parameter values $(\varrho, \lambda, \gamma)$ satisfying (ii) and (28) is non-empty if $0 < \varrho < 1/2$.

Condition (ii) and (28) are sufficient conditions for h_I to be an optimal policy function that generates chaotic dynamics. There may be other sufficient conditions. In fact, Nishimura and Yano [31,32] provide an alternative and constructive method to find parameter values (λ, γ) for which h_I describes optimal paths that are ergodically chaotic and geometrically sensitive. This method works for any given discount factor ϱ, even if it is arbitrarily close to 1. Nishimura, Sorger, and Yano [29] extend the results of Nishimura and Yano [31,32] to the case in which the objective function is strictly concave.

[8] A function h is called expansive if it is piecewise differentiable with $|h'(x)| > 1$ for all x at which h is differentiable. For unimodal and expansive maps, we can use the results of Lasota and Yorke [20] and Li and Yorke [22] to exhibit ergodic chaos.

Finally, Nishimura, Shigoka, and Yano [28] exhibit a model with differentiable production functions and a dynamical system that is topologically chaotic for ϱ arbitrarily close to 1, and contains that from Nishimura and Yano [31,32] as a limiting case.

5 Stochastic growth models

In the remainder of the paper, the one-sector optimal accumulation model of Section 2 is again examined, but this time in the presence of uncertainty resulting from a stochastic production relationship. First the problem is cast in the framework of stochastic dynamic programming. Asymptotic properties of the resulting Markov process are then considered.

5.1 Notation

As in the deterministic one-sector case, the state space for the model is $[0, \infty)$. When taken as a measurable space, $[0, \infty)$ is always associated with its Borel sets $\mathscr{B}$. Let B_b be the bounded Borel functions on $[0, \infty)$, let C_0 be the set of continuous functions with compact support, and let C_b be the collection of continuous bounded functions. All three spaces are endowed with the usual sup norm. [9] Also, let $\mathscr{M}$ be the set of finite signed Borel measures on $[0, \infty)$.

By the Riesz-Radon theorem, $\mathscr{M}$ is isomorphic to the norm dual of C_0. When considered as a topological space, $\mathscr{M}$ is given the so-called narrow topology induced by C_b. [10] Let $\mathscr{P}$ be the set of all $\mu \in \mathscr{M}$ such that $\mu \geq 0$ and $\mu[0, \infty) = 1$. Here $\mathscr{P}$ inherits the relative topology. Elements of $\mathscr{P}$ are associated one-for-one with the distribution functions on $[0, \infty)$. Distributions concentrated at a point x are denoted δ_x.

In the sequel, Markov processes are constructed as follows. A stochastic difference equation

$$x_{t+1} = h(x_t, \varepsilon_t), \quad h\colon X \times X \to X \tag{29}$$

is given. Here the shocks ε_t are assumed to be uncorrelated and identically distributed by $\psi \in \mathscr{P}$. The function h is appropriately measurable. From (29), a Markov kernel

$$X \times \mathscr{B} \ni (x, B) \mapsto N(x, B) := \int_X \mathbf{1}_B[h(x, z)]\psi(dz) \in [0, 1] \tag{30}$$

[9] That is, $\|f\| := \sup_{x \geq 0} |f(x)|$.
[10] Elements of C_b provide real functions on $\mathscr{M}$ by the natural inner product. The narrow topology is the weakest topology that makes all such functions continuous.

is obtained. As usual, $N(x, B)$ is interpreted as the conditional probability of traveling from x into B in one step. From the kernel N and a starting point $x \in [0, \infty)$ the canonical Markov process (x_t) for the state variables on the sequence space $[0, \infty)^{\mathbb{N}}$ can be constructed [39, Theorem II.9.2]. Let its distribution be denoted $\mathbf{P}_x$. Also, let the marginal distributions of each x_t be given by $\varphi_t \in \mathscr{P}$. That is, $\varphi_t B = \mathbf{P}_x\{x_t \in B\}$. It is well-known that the sequence (φ_t) satisfies the recursion $\varphi_{t+1} = P\varphi_t$, where P denotes the operator

$$(P\varphi)(B) := \int N(x, B)\varphi(dx) \tag{31}$$

from $\mathscr{P}$ into itself. Define also the operator dual to P by $U : B_b \to B_b$,

$$(Uv)(x) := \int v(y)N(x, dy). \tag{32}$$

Here U and P are said to be dual because

$$\langle v, P\mu\rangle = \langle Uv, \mu\rangle, \quad \forall v \in B_b,\ \mu \in \mathscr{P}, \tag{33}$$

where the inner-product binary of course means integration.

If P^t is the t-th composition of P with itself, then clearly $\varphi_t = P^t\varphi_0$. The operator P is called the Markov operator (or stochastic operator, or propagator) associated with (29).

5.2 *Stochastic optimal growth*

The essential difference from the one-sector model of Section 2 is that production is no longer deterministic. Uncertainty in production may arise from a number of sources, such as weather and other natural phenomena, or exogenous changes in input-output relationships (new technologies, etc.). Suppose in particular that

$$Y_{t+1} = F(K_t, L_t)\varepsilon_t. \tag{34}$$

The shocks ε_t are temporally uncorrelated and identically distributed according to $\psi \in \mathscr{P}[0, \infty)$. Regarding F and its intensive form f the assumptions of Section 2 are maintained.

For the stochastic case it proves convenient to modify slightly the timing of the problem and the state variable. The sequence of events is as follows. Let t be the current time. A current level of income y_t is observed. Subsequently, a level of consumption c_t and therefore savings is chosen in $[0, y_t]$. Savings determines current capital stock k_t available for production; in fact we take savings and capital stock to be equal ($\delta = 1$). Next, the shock ε_t is drawn by "nature," and next period income y_{t+1} is realized via (34). The process then repeats.

Social welfare is optimized by solving

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \varrho^t u(c_t)\right] \tag{35}$$

subject to the constraint

$$k_t + c_t = y_t, \quad y_{t+1} = f(k_t)\varepsilon_t. \tag{36}$$

This problem can be conveniently treated within the framework of stochastic dynamic programming. In the infinite horizon case the problem is stationary, and hence we seek a fixed control policy g that associates to current income y_t a feasible consumption $c_t = g(y_t) \leq y_t$.

For each such g, substituting into (36) gives the stochastic difference equation

$$y_{t+1} = f[y_t - g(y_t)]\varepsilon_t. \tag{37}$$

From (37) a Markov kernel is determined via (30), and hence a distribution $\mathbf{P}_y^g$ for the canonical process over the space of real nonnegative sequences. Here y is the initial level of income, and the superscript g recalls the dependence of the distribution on the particular policy chosen. The value of the policy for social welfare can now be assessed by calculating (35), where the expectation is taken with respect to $\mathbf{P}_y^g$ and $c_t = g(y_t)$.

To further integrate the current topic with the standard theory of stochastic dynamic programming, assume also that

Assumption 5 *The utility function u is bounded and nonnegative.*[11]

5.3 Properties of the optimal policy

Beginning with Brock and Mirman [6], the properties of the optimal policy for this model have been extensively investigated. The first step is to obtain Bellman's optimality equation, which is given below. Full proofs of the following results are available in, for example, Harris [14].

[11] Relaxing this assumption in the context of this model and its variants continues to be a very active research area. See, for example, Durán [12].

Theorem 5 *There exists a unique $v^* \in B_b$ satisfying*

$$v^*(y) = \max_{0 \le c \le y} \left\{ u(c) + \varrho \int v^*[f(y-c)z]\psi(dz) \right\}. \tag{38}$$

This function v^* is continuous, strictly increasing and strictly concave.

This is the familiar Bellman optimality condition, where v^* is the value function. An outline of the proof is as follows. Define a self-mapping $T \colon B_b \to B_b$ by

$$Tv(y) := \sup_{0 \le c \le y} \left\{ u(c) + \varrho \int v[f(y-c)z]\psi(dz) \right\}. \tag{39}$$

By the standard theory [14, Theorem 2.1] the operator T can be shown to be uniformly contracting on B_b, indicating a unique fixed point v^* in that space, and that $T^n v \to v^*$ in sup norm for any bounded v. Also, in the present case, it can be shown using the continuity assumptions and the theorem of the maximum that $TC_b \subset C_b$, in which case v^* as the limit of such functions must be continuous. Further, concavity and monotonicity assumptions on u and f imply that T maps increasing, concave functions into strictly increasing, strictly concave functions. Since v^* can be written as the limit of such functions it must be increasing and concave, and since $Tv^* = v^*$, the properties are in fact strict.

Clearly, a unique, single valued solution $g(y)$ to the right hand side of (38) exists for each y. Applying the theorem of the maximum shows that g is in fact continuous. The standard theory of dynamic programming now indicates that this function is the (unique) optimal policy that maximizes the expectation of discounted utilities in (35).

One of the distinguishing features of economic systems is that diminishing returns for both consumption and technology in many cases seem appropriate, and these in turn lead directly to convexity. The latter is of course closely related to differentiability. Indeed, Benveniste and Scheinkman [2] use convexity to establish that

Theorem 6 *The value function v^* is differentiable on the interior of its domain. Moreover,*

$$\frac{dv^*(y)}{dy} = u'[g(y)]. \tag{40}$$

The condition (40) is an envelope condition. For a derivation see, for example, Mirman and Zilcha [24, Lemma 1].

Derivative conditions typically provide sharp characterizations of the solution to optimization problems. The present case is no exception. For example, it is straightforward to show from (40) and the established properties of v^* that the function g is nondecreasing, so that consumption increases with higher income. In fact the same is true for savings. In summary,

Lemma 7 *The optimal policy g is continuous. Both $y \mapsto g(y)$ and $y \mapsto y - g(y)$ are nondecreasing.*

5.4 Dynamics

In the stochastic case, invariant distributions for the associated Markov process have a natural interpretation as long run equilibria. At such distributions the probabilities of different outcomes become fixed. Economic policy can be evaluated in part by weighing the effects of various parameters on the invariant distributions and their moments.

Invariant distributions are fixed points of the operator P defined in (31) on the space $\mathscr{P}[0,\infty)$. To justify their role as focal points for long run outcomes, it is desirable in addition that these invariant distributions have some kind of stability property. Of particular interest is the case where the fixed point is unique and asymptotically stable. That is, $P^t\varphi$ converges to this limit as $t \to \infty$ for all initial $\varphi \in \mathscr{P}$. Economies with this property are called history independent.

Brock and Mirman [6] first proved the stability result stated above for the one-sector stochastic growth model in a topology stronger than the narrow topology. They assume that the shock ε has compact support. Their techniques were rather specific to the problem in question, however. In what follows, we outline proofs for a number of similar results that are perhaps more easily generalizable to other related economic models.

Following Brock and Mirman, the great majority of research—see Stachurski [43] for further references—has assumed that the shock has compact support:

Assumption 6 *There exist numbers $0 < a \leq b < \infty$ such that $\psi[a,b] = 1$.*

A preliminary result is that

Lemma 8 *If Assumption 6 holds then the economy defined by u, f ϱ and ψ has an invariant distribution.*

We outline a proof, which uses only decreasing returns. By Assumption 2 clearly there exists a unique positive number p such that $p = f(p)b$, and (37) indicates that when $y_t \in [0,p]$, $y_{t+1} \in [0,p]$ with probability one. From this it can be shown via (30) and (31) that P is invariant on that subset $\mathscr{P}_0$ of all $\varphi \in \mathscr{P}$ with $\varphi[0,p] = 1$. This set $\mathscr{P}_0$ is tight and closed. It follows from

Prokhorov's Theorem [11, Theorem 11.5.4] that $\mathscr{P}_0$ is compact in the narrow topology. Evidently it is convex. Further, P is continuous in this topology as a result of continuity of g and f. An application of the Markov-Kakutani fixed point theorem now gives the desired result.

5.5 Stochastic monotonicity

This result can be improved on in a number of ways. First, uniqueness and stability are not established. Second, it is desirable to show that at least one invariant measure is not concentrated at zero, meaning that the economy can operate in the long run at positive levels of income. In other words, the economy does not collapse as a result of excessive consumption and negative shocks. Third, one would like to relax Assumption 6 in order to integrate the model better with standard econometric treatments.

In this connection, an interesting result of Hopenhayn and Prescott [16] is now presented. Their method solves the first and second questions raised above: they demonstrate the existence of a unique non-zero invariant measure and asymptotic convergence from all initial conditions. The paper is interesting from a number of other perspectives. First, their techniques are more constructive than simply invoking the fixed point theorem of Markov-Kakutani. Second, they do not rely on continuity. Continuity played a key part in the proof of Lemma 8 given above, but it is not clear why this property should be an obvious consequence of rational agent behavior. Indeed, it has often been argued that models of discontinuous optimal behavior have better potential for representing economic time series.

Theorem 9 *If Assumption 6 holds then the economy defined by u, f ϱ and ψ has a unique invariant distribution φ^* distinct from δ_0, and $P^t\varphi \to \varphi^*$ as $t \to \infty$ for every initial condition $\varphi \in \mathscr{P}$.*

The proof runs as follows. First, it is observed that "monotone" Markov processes on a compact set have at least one invariant distribution (definitions are given below). Also, these hypotheses combined with a mixing condition are demonstrated to imply uniqueness of the invariant distribution and asymptotic stability. Finally, the authors show that the model in question is appropriately monotone and mixing on a compact set, and that the invariant distribution is not concentrated at zero.

We prove only existence. For this purpose, define a partial order $\preccurlyeq$ on $\mathscr{P}$ by so-called stochastic dominance. That is, $\mu \preccurlyeq \mu'$ if and only if $\int v d\mu \leq \int v d\mu'$ for every nonnegative and nondecreasing $v \in B_b$.[12] As usual, a self-mapping

[12] Equivalently, if the distribution function associated with μ lies entirely above that

T on $(\mathscr{P}, \preccurlyeq)$ is called nondecreasing if $\mu \preccurlyeq \mu'$ implies $T\mu \preccurlyeq T\mu'$.

Observe that the Markov operator P associated with the economy in question is nondecreasing in this order. The reason is that savings is nondecreasing in income (Lemma 7), and hence $y \mapsto f[y - g(y)]$ is likewise nondecreasing. It follows that for any nonnegative, nondecreasing $v \in B_b$ and any pair $\mu, \mu' \in \mathscr{P}$, $\mu \preccurlyeq \mu'$,

$$\begin{aligned}\langle v, P\mu\rangle &= \langle Uv, \mu\rangle \\ &= \int \left[\int v[f(y - g(y))z]\psi(dz)\right] \mu(dy) \\ &\leq \int \left[\int v[f(y - g(y))z]\psi(dz)\right] \mu'(dy) = \langle v, P\mu'\rangle.\end{aligned}$$

Since P is nondecreasing everywhere it is nondecreasing on the compact set $\mathscr{P}_0$ defined above. As before, P is invariant on the latter: $P\mathscr{P}_0 \subset \mathscr{P}_0$. Since every chain in $(\mathscr{P}_0, \preccurlyeq)$ has a supremum [16, Proposition 1], and since $(\mathscr{P}_0, \preccurlyeq)$ has a least element δ_0, an invariant distribution for P exists by the Knaster-Tarski fixed point theorem.

5.6 Noncompact state space

To date almost all treatments have maintained the assumption that the production shock has compact support. Indeed this assumption is crucial to both proofs discussed above. However, ideally the model should be amenable for empirical testing and prediction, and econometrics typically deals with standard shocks from mathematical statistics. Recently Stachurski [43] has shown that the one-sector growth model is asymptotically stable for many such shocks.

Theorem 10 *Let the distribution of the shock ε be absolutely continuous with respect to Lebesgue measure. If the representative density ψ is positive almost everywhere on $[0, \infty)$ and satisfies $\int x\psi(x)dx < \infty$, $\int (1/x)\psi(x)dx < 1$, then the one-sector model has a unique invariant distribution. This invariant distribution is globally attracting in the norm topology.*

Absolute continuity appears to be critical here, because the Markov operator then maps probability measures into absolutely continuous probability measures, a subset of L_1. New techniques for dealing with Markov processes in L_1 have been introduced by Lasota [19]. In this framework, the above result is verified by showing that the Markov operator is strongly contracting in norm distance, and that every trajectory under P is norm precompact. Together these properties imply the stated result [43, Theorem 5.2].

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of μ'.

The question of whether the same norm convergence still holds without absolute continuity is open.

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