On Geometric Ergodicity of the Commodity Pricing Model *

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Abstract

We provide a simple proof of geometric ergodicity for Samuelson's commodity pricing model (1971, *Proc. Nat. Acad. Sci.*, 68, 335–337). The proof yields a rate of convergence to the stationary distribution stated in terms of model primitives. We also provide a rate of convergence for prices to the stationary price process, and for the joint distribution of the state process to the stationary state process.

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1 Introduction

We study dynamics of the commodity pricing model introduced by Samuelson (1971). These dynamics were investigated by Scheinkman and Schectman (1983), who confirmed a conjecture of Samuelson that the state variable is stationary and ergodic. Bobenrieth, Bobenrieth and Wright (2002) show that the model is in fact geometrically ergodic. Their proof uses function-analytic methods, based on quasicompactness and equicontinuity of the Markov operator.

In this paper we give a simple and direct proof of geometric ergodicity which requires only elementary arguments. In addition, the technique gives an upper bound on the *rate* of convergence to the stationary distribution stated in terms of the primitives of the model. Further, we provide computable bounds on the deviation of the joint distribution of the entire stochastic process from the joint stationary distribution; and the distribution of the price process from the stationary price process.

2 Formulation of the Problem

Our benchmark commodity pricing model is the model studied in Deaton and Laroque (1992). In this section we briefly state the main features of the model. The market is for a single commodity, the "harvest" of which is an IID process $(\xi_t)_{t \geq 1}$ on $[0, b]$ with cumulative distribution function ϕ. We assume that if $z > 0$, then $\phi(z) := \mathbb{P}\{\xi_t \leq z\} > 0$.[1] A storage technology permits transfer of the commodity from the current period to the next. Storage costs are positive: Quantity I_{t-1} carried over from $t-1$ yields γI_{t-1} at t, where γ lies in $(0, 1)$.

[1] Each ξ_t is defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Demand for the commodity is the sum of demand from speculators and from consumers. Consumer demand is determined according to a fixed demand schedule, while speculator demand depends on present and expected future prices. Supply at time t is the sum of the harvest ξ_t and the depreciated carryover γI_{t-1}. Thus, if X_t denotes supply, then $X_t = \gamma I_{t-1} + \xi_t$. Speculators now purchase a quantity $I_t \in [0, X_t]$, consumers purchase a quantity C_t, and the process repeats.

It can be shown that the equilibrium price p_t that equates supply and demand for the commodity at time t can be represented as a stationary function of the state X_t. That is, $p_t = p(X_t)$ for some measurable function p. Using the equilibrium conditions, this pricing function is naturally defined as the fixed point of a self-mapping on a certain function space. The details do not concern us here, and interested readers are referred to Deaton and Laroque (1992).

Similarly, equilibrium investment for this model can be shown to be a function I of the state variable; that is, $I_t = I(X_t)$ for all t. Given this function, the process for the state is then

$$X_{t+1} = \gamma I(X_t) + \xi_{t+1}, \quad (\xi_t)_{t \geq 0} \stackrel{\text{IID}}{\sim} \phi, \quad X_0 \text{ given} \tag{1}$$

If $\bar{s} := (1-\gamma)^{-1}b$ and if $X_{t-1} \in S := [0, \bar{s}]$ then one can show that $X_t \in S$, so that $S = [0, \bar{s}]$ is a valid state space for the process $(X_t)_{t \geq 0}$. Furthermore, one can show that I is monotone nondecreasing, and that there exists an $x_b > 0$ such that $x \leq x_b$ implies $I(x) = 0$.[2]

If we define F_z to be the map $x \mapsto F_z(x) := \gamma I(x) + z$, then X_t can be written as as

$$X_t = F_{\xi_t} \circ F_{\xi_{t-1}} \circ \cdots \circ F_{\xi_1}(X_0)$$

A distribution ψ^* is called stationary for this process if it satisfies

$$\psi^*(B) = \int \left[\int \mathbb{1}_B[F_z(x)] \phi(dz) \right] \psi^*(dx) \qquad (B \in \mathscr{B}(S)) \tag{2}$$

Here $\mathbb{1}_B$ denotes the indicator function of B, while $\mathscr{B}(S)$ is the Borel subsets of S. It is well known that such a distribution exists.[3]

Letting $\mathscr{L}Y$ denote the distribution (or law) of any given random variable Y, the stationary distribution ψ^* has the property that if $\mathscr{L}X_0 = \psi^*$, then $\mathbf{X} := (X_t)_{t \geq 0}$ is stationarity. In particular, $\mathscr{L}X_t = \mathscr{L}X_0 = \psi^*$ for all $t \in \mathbb{N}$. Even if $\mathscr{L}X_0 \neq \psi^*$, Bobenrieth et al. (2002) show that $\mathscr{L}X_t \to \psi^*$ as $t \to \infty$

[2] Again, details and proofs can be found in Deaton and Laroque (1992). Throughout the paper we assume that X_0 is independent of $(\xi_t)_{t \geq 1}$.

[3] This follows from the monotonicity (or continuity) of I and the compactness of S—see, e.g., Hopenhayn and Prescott (1992, Corollary 2).

at a geometric rate. The metric used in their analysis is the total variation (TV) distance, which for probability measures μ and ν on S is defined as

$$\|\mu - \nu\| := \sup\{|\mu(B) - \nu(B)| : B \in \mathscr{B}(S)\}$$

TV convergence is considerably stronger than the classical notion of convergence in distribution.[4]

3 Geometric Ergodicity

We now present the key mathematical result of the paper, which includes a simple direct proof of geometric ergodicity for the quantity process $(X_t)_{t \geq 0}$, and, more importantly, provides a rate of convergence stated in terms of model primitives.

Theorem 3.1. *If $z_0 \in [0, b]$ and $k \in \mathbb{N}$ are chosen such that*

$$\gamma^k \bar{s} + z_0 \frac{1 - \gamma^k}{1 - \gamma} \leq x_b \tag{3}$$

[4]Indeed, if $\mu_n \to \mu$ in total variation and F_n and F are the respective distribution functions on S, then $F_n \to F$ uniformly on S. TV distance has the following highly quantitative interpretation: If we are approximating the stationary distribution ψ^* by the computable distribution ψ, and if one can show that $\|\psi^* - \psi\| \leq \epsilon$, then for any event B one has $\psi(B) - \epsilon \leq \psi^*(B) \leq \psi(B) + \epsilon$.

then for any initial condition X_0 we have

$$\|\mathscr{L} X_t - \psi^*\| \leq (1 - \phi(z_0)^k)^{\lfloor \frac{t-1}{k} \rfloor} \qquad (t \in \mathbb{N}) \tag{4}$$

In the statement of the theorem, the notation $\lfloor a \rfloor$ refers to the largest integer smaller than a. The bound (4) can be alternatively written as

$$\|\mathscr{L}[F_{\xi_t} \circ \cdots \circ F_{\xi_1}(X_0)] - \psi^*\| \leq (1 - \phi(z_0)^k)^{\lfloor \frac{t-1}{k} \rfloor} \qquad (t \in \mathbb{N}) \tag{5}$$

Convergence of this bound to zero requires $\phi(z_0) > 0$. One can always choose such a z_0.[5]

Remark. Note that (4) holds for *any* pair z_0 and k satisfying (3). This pair can be selected to minimize the right hand side of (4) given t.

In applications we are often more interested in the dynamics of prices, rather than quantities. The following corollary extends the quantity result in Theorem 3.1 to prices. In the corollary, $\mathbf{X} = (X_t)_{t \geq 0}$ is an arbitrary equilibrium process with initial condition X_0 and $\mathbf{X}^* = (X_t^*)_{t \geq 0}$ is the stationary process with $\mathscr{L} X_0^* = \psi^*$.

[5]In view of the fact that $0 < x_b$, a sufficiently large k and sufficiently small $z_0 > 0$ satisfy (3). By our assumptions on the harvest, $z_0 > 0$ implies $\phi(z_0) > 0$.

Corollary 3.1. *Let $p_t = p(X_t)$ and let p_t^* be the stationary price process defined by $p_t^* = p(X_t^*)$. Then for z_0 and k as in Theorem 3.1,*

$$\|\mathscr{L} p_t - \mathscr{L} p_t^*\| \leq (1 - \phi(z_0)^k)^{\lfloor \frac{t-1}{k} \rfloor} \qquad (t \in \mathbb{N})$$

Proof. Let $S_P := p(S)$, the set of points in which prices take values, and let $\mathscr{B}(S_P)$ be the Borel subsets. Pick any $B \in \mathscr{B}(S_P)$. We have

$$\begin{aligned} |\mathbb{P}\{p_t \in B\} - \mathbb{P}\{p_t^* \in B\}| &= |\mathbb{P}\{p(X_t) \in B\} - \mathbb{P}\{p(X_t^*) \in B\}| \\ &= |\mathbb{P}\{X_t \in p^{-1}(B)\} - \mathbb{P}\{X_t^* \in p^{-1}(B)\}| \end{aligned}$$

As p is measurable we have $p^{-1}(B) \in \mathscr{B}(S)$.

$$\therefore \quad |\mathbb{P}\{p_t \in B\} - \mathbb{P}\{p_t^* \in B\}| \leq \sup_{A \in \mathscr{B}(S)} |\mathbb{P}\{X_t \in A\} - \mathbb{P}\{X_t^* \in A\}|$$

$$\therefore \quad |\mathbb{P}\{p_t \in B\} - \mathbb{P}\{p_t^* \in B\}| \leq (1 - \phi(z_0)^k)^{\lfloor t-1/k \rfloor}$$

Taking the supremum over all $B \in \mathscr{B}(S_P)$ and using the definition of the total variation norm establishes the statement in the lemma. □

In the remainder of this section we discuss the intuition behind the proof of Theorem 3.1. Recall the following "coupling" inequality (cf., e.g., Lindvall, 1992), which states that if the probability X and Y differ is small, then so is the distance between their laws.

Lemma 3.1. *If X and Y are any two random variables on common probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathscr{L}X = \mu$ and $\mathscr{L}Y = \nu$, then*

$$\|\mu - \nu\| \leq \mathbb{P}\{X \neq Y\} \tag{6}$$

The beauty of the lemma is that (6) holds for any X and and Y with $\mathscr{L}X = \mu$ and $\mathscr{L}Y = \nu$. Careful choice of X and Y can lead to a tight bound. To illustrate, let $\mathbf{X} = (X_t)_{t\geq 0}$ be a given equilibrium process starting at arbitrary X_0, and let $\mathbf{X}^* = (X_t^*)_{t\geq 0}$ be the stationary process. Crucially, we assume that both are driven by identical harvests $(\xi_t)_{t\geq 1}$. Specifically

$$X_t = F_{\xi_t} \circ \cdots \circ F_{\xi_1}(X_0) \quad \text{and} \quad X_t^* = F_{\xi_t} \circ \cdots \circ F_{\xi_1}(X_0^*)$$

By $\mathscr{L}X_t^* = \psi^*$ and (6) it follows immediately that

$$\|\mathscr{L}X_t - \psi^*\| = \|\mathscr{L}X_t - \mathscr{L}X_t^*\| \leq \mathbb{P}\{X_t \neq X_t^*\} \quad \forall\, t \in \mathbb{N} \tag{7}$$

Thus to bound $\|\mathscr{L}X_t - \psi^*\|$ it is sufficient to bound $\mathbb{P}\{X_t \neq X_t^*\}$. In other words, we need to show that the probability X_t and X_t^* remain distinct converges to zero in t—or, conversely, that X_t and X_t^* are eventually equal with high probability. Although the state space is uncountable, which makes it challenging to show that X_t and X_t^* are eventually exactly equal with high probability, there are two features of our set up which make the approach feasible:

Property 1. $\mathbf{X}$ and $\mathbf{X}^*$ are driven by the *same* sequence of harvests $(\xi_t)_{t\geq 1}$. As a result, if $X_j = X_j^*$ for some j, then $X_t = X_t^*$ for all $t \geq j$.

Property 2. If both $X_j \leq x_b$ and $X_j^* \leq x_b$, then $I(X_j) = I(X_j^*) = 0$ and hence $X_{j+1} = X_{j+1}^* = \xi_{j+1}$.

As a consequence of these two properties, for $X_t = X_t^*$ to hold it is sufficient that both $X_j \leq x_b$ and $X_j^* \leq x_b$ for some $j < t$. This will occur whenever there is a sufficiently long sequence (a sequence of length k, say) of sufficiently small harvests (i.e., below some value z_0). As an illustration, Figure 1 shows simulated paths for the two time series $\mathbf{X}$ and $\mathbf{X}^*$. At $t = 4$ both X_t and X_t^* are below x_b. As a result, $X_t = X_t^*$ for all $t \geq 5$. The two processes are said to couple at $t = 5$, and that date is referred as the coupling time.

Since $(\xi_t)_{t\geq 1}$ is IID, the probability that a sequence of harvest sufficiently poor to force $X_j \leq x_b$ and $X_j \leq x_b$ has occurred at least once prior to t converges to 1 as $t \to \infty$. As a result, $\mathbb{P}\{X_t \neq X_t^*\}$ converges to zero. The remainder of the proof makes this argument more precise, and is deferred to the appendix.

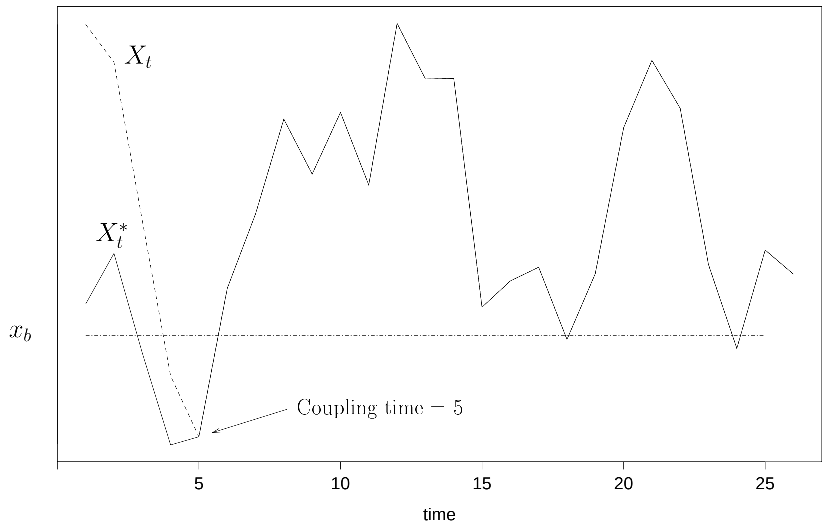

Figure 1: Coupling of (X_t) and (X_t^*) at $t = 5$

4 Joint Distributions

Finally we establish a bound on the deviation between the *joint* distribution of the process starting at arbitrary ψ and the joint stationary distribution. To state the result, let $\mathscr{B}(S^{\mathbb{N}})$ be the product σ-algebra on $S^{\mathbb{N}}$. Then

$$\sup_{U \in \mathscr{B}(S^{\mathbb{N}})} |\mathbb{P}\{(X_t, X_{t+1}, \ldots) \in U\} - \mathbb{P}\{(X_t^*, X_{t+1}^*, \ldots) \in U\}|$$
$$\leq \mathbb{P}\{(X_t, X_{t+1}, \ldots) \neq (X_t^*, X_{t+1}^*, \ldots)\}$$

by Lemma 3.1. But since $X_t = X_t^*$ implies $X_j = X_j^*$ for all $j \geq t$ we have

$$\{(X_t, X_{t+1}, \ldots) = (X_t^*, X_{t+1}^*, \ldots)\} = \{X_t = X_t^*\}$$

$$\therefore \quad \{(X_t, X_{t+1}, \ldots) \neq (X_t^*, X_{t+1}^*, \ldots)\} = \{X_t \neq X_t^*\}$$

We can state these relations more succinctly using the left shift operator θ. As before, let $\mathbf{X} := (X_t)_{t \geq 0}$ and $\mathbf{X}^* := (X_t^*)_{t \geq 0}$. Further, let $\theta \colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ be the left shift, so that

$$\theta \mathbf{X} = (X_1, X_2, \ldots), \quad \theta^t \mathbf{X} = (X_t, X_{t+1}, \ldots),$$

and so on. In this notation, we have established that

$$\sup_{U \in \mathscr{B}(S^{\mathbb{N}})} |\mathbb{P}\{\theta^t \mathbf{X} \in U\} - \mathbb{P}\{\theta^t \mathbf{X}^* \in U\}| \leq \mathbb{P}\{\theta^t \mathbf{X} \neq \theta^t \mathbf{X}^*\}$$

and $\{\theta^t \mathbf{X} \neq \theta^t \mathbf{X}^*\} = \{X_t \neq X_t^*\}$. As a result,

$$\sup_{U \in \mathscr{B}(S^{\mathbb{N}})} |\mathbb{P}\{\theta^t \mathbf{X} \in U\} - \mathbb{P}\{\theta^t \mathbf{X}^* \in U\}| \leq \mathbb{P}\{X_t \neq X_t^*\}$$

$$\therefore \quad \sup_{U \in \mathscr{B}(S^{\mathbb{N}})} |\mathbb{P}\{\theta^t \mathbf{X} \in U\} - \mathbb{P}\{\theta^t \mathbf{X}^* \in U\}| \leq (1 - \phi(z_0)^k)^{\lfloor t-1/k \rfloor}$$

Finally, since $\mathbf{X}^*$ is stationary the left shift θ is invariant in the sense that both $\mathbf{X}^*$ and $\theta^t \mathbf{X}^*$ have the same distribution, so we obtain

Lemma 4.1. *For any equilibrium process $\mathbf{X}$ with arbitrary initial condition X_0 we have*

$$\|\mathscr{L}(\theta^t \mathbf{X}) - \mathscr{L}\mathbf{X}^*\| \leq (1 - \phi(z_0)^k)^{\lfloor \frac{t-1}{k} \rfloor}$$

for all $t \in \mathbb{N}$, where k and z_0 are as defined in Theorem 3.1.

This is stronger than our original bound in Theorem 3.1 as it clearly implies the latter: by setting $U = B \times S \times S \times \cdots$ one recovers

$$|\mathbb{P}\{X_t \in B\} - \psi^*(B)| \leq (1 - \phi(z_0)^k)^{\lfloor t-1/k \rfloor}$$

5 Appendix

Proof of Theorem 3.1. We now complete the proof of Theorem 3.1. To check for occurrences of the event $\{X_j \leq x_b \text{ and } X_j^* \leq x_b\}$, we define a third process which acts as an upper bound for (X_t) and (X_t^*):

$$X'_{t+1} = \gamma X'_t + \xi_{t+1}, \quad X'_0 = \bar{s}$$

Thus, (X'_t) is the process for the state when all of the harvest is carried over in each state, and the initial state is $\bar{s}$. As $X'_0 = \bar{s}$ and $I(x) \leq x$ for all $x \in S$, it follows that (X'_t) dominates both (X_t) and (X_t^*). Hence to check $X_t \leq x_b$ and $X_t^* \leq x_b$ it is sufficient to check $X'_t \leq x_b$.

Given that if (X_t) and (X_t^*) meet they remain equal, and given that $X'_j \leq x_b$ implies $X_{j+1} = X_{j+1}^*$, it must be the case that

$$X'_j \leq x_b \text{ for some } j \leq t \implies X_{t+1} = X_{t+1}^*$$

$$\therefore \quad \mathbb{P}\{X'_j \leq x_b \text{ for some } j \leq t\} \leq \mathbb{P}\{X_{t+1} = X^*_{t+1}\}$$

$$\therefore \quad \mathbb{P}\{X_{t+1} \neq X^*_{t+1}\} \leq \mathbb{P} \cap_{j \leq t} \{X'_j > x_b\}$$

The probability of the event $\cap_{j \leq t}\{X'_j > x_b\}$ can be bounded relatively easily. Indeed, suppose that harvests ξ_1 to ξ_k are all below z_0, where k and z_0 are as in the statement of the theorem (i.e., chosen to satisfy (3)). Then

$$X'_j \leq \gamma X'_{j-1} + z_0, \quad j = 1, \ldots, k$$

Combining these k inequalities gives

$$X'_k \leq \gamma^k X'_0 + z_0 \frac{1-\gamma^k}{1-\gamma}$$

$$\therefore \quad X'_k \leq \gamma^k \bar{s} + z_0 \frac{1-\gamma^k}{1-\gamma} \leq x_b$$

where the second inequality follows from (3). Thus, a sequence of k harvests below z_0 forces (X'_t) below x_b by the end of the sequence.

In the preceding argument we considered the sequence $\xi_1, \ldots, \xi_k$. The same logic clearly works for any k consecutive harvests, irrespective of the date: If k *consecutive* harvests fall below z_0 during the period prior to t inclusive, then $X'_j \leq x_b$ for some $j \leq t$.

In the time from period 1 to period t there are precisely $\lfloor t/k \rfloor$ nonoverlapping sequences of k consecutive harvests. Let E_i be the event that the i-th of these sequences has all harvests below z_0. That is,

$$E_i = \cap_{j=k\times(i-1)+1}^{i\times k} \{\xi_j \leq z_0\}$$

If one event E_i occurs, then the dominating process (X_t') satisfies $X_j' \leq x_b$ for some $j \leq t$. Put differently, if X_j' never falls below x_b in the period up to t, then none of the events E_i has occurred.

$$\therefore \quad \mathbb{P} \cap_{j\leq t} \{X_j' > x_b\} \leq \mathbb{P} \cap_{i=1}^{\lfloor t/k \rfloor} E_i^c$$

Since the sequences of harvests that make up each E_i are nonoverlapping these events are independent. It follows that

$$\mathbb{P} \cap_{i=1}^{\lfloor t/k \rfloor} E_i^c = \prod_{i=1}^{\lfloor t/k \rfloor} (1 - \mathbb{P}(E_i))$$

Evidently $\mathbb{P}(E_i) = \phi(z_0)^k$, from which we obtain

$$\prod_{i=1}^{\lfloor t/k \rfloor} (1 - \phi(z_0)^k) = (1 - \phi(z_0)^k)^{\lfloor t/k \rfloor}$$

$$\therefore \quad \mathbb{P}\{X_{t+1} \neq X_{t+1}^*\} \leq (1 - \phi(z_0)^k)^{\lfloor t/k \rfloor}$$

In view of (7) we have the desired inequality

$$\|\mathscr{L} X_{t+1} - \psi^*\| \leq (1 - \phi(z_0)^k)^{\lfloor t/k \rfloor}$$

□

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